Discrete Choice, Complete Markets, and Equilibrium

Simon Mongey
Federal Reserve Bank of Minneapolis and NBER

Michael E. Waugh
Federal Reserve Bank of Minneapolis and NBER

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ABSTRACT

This paper characterizes the allocations that emerge in general equilibrium economies populated by households with preferences of the additive random utility type that make discrete consumption, employment or spatial decisions. We start with a complete markets economy where households can trade claims contingent upon the realizations of their preference shocks. We (i) establish a first and second welfare theorem, (ii) illustrate that in the absence of ex-ante trade, discrete choice economies are generically inefficient, (iii) show that complete markets are not necessary and a much smaller set of securities decentralizes the efficient allocation. We illustrate the relevance of these results in several canonical settings and for measuring how welfare changes in response to changes in prices.

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In this paper, we study the efficiency properties of competitive equilibria of economies populated by households with preferences of the form

$$u^i(c^i_j) + \xi^i_j,$$

where j is a finite set of goods / locations / occupations and ξ^i_j is an idiosyncratic random variable specific to household i. *Discreteness* is the restriction that households can choose only one type of good to consume, location to live in, occupation to work in, and so on.

To say these models have proven to be useful in many applications is an understatement. They are a foundation for modeling consumer demand see, for example, McFadden (1974), Anderson et al. (1992) and Berry et al. (1995). In spatial settings, preferences of this form have proven useful for studying location and migration decisions, as in Kennan and Walker (2011), Kline and Moretti (2014), Diamond (2016), Redding and Rossi-Hansberg (2017). This setting also connects with studies of labor supply and amenity-driven occupational choice, like Rogerson (1988), Card et al. (2018), Berger et al. (2022). They have been used in static and dynamic contexts.

These models are often used to evaluate the welfare impacts of various counterfactual policy scenarios. But what are properties of the allocations from which welfare is evaluated? Do welfare theorems hold in these economies? If they don't, what market instruments are necessary to achieve Pareto efficient allocations? When prices change — say, because of technological or policy interventions — how does efficiency (or lack thereof) affect the welfare impacts?

The first part of the paper describes a general choice framework. We consider a class of utility functions that allows us to connect with how these models have been used in studies of consumer demand, spatial economics, or labor supply. We don't appeal to functional forms on the distribution over the taste shocks ξ^i_j or utility function $u^i(\cdot)$. Thus, our results are not specific to, say, the popular type 1 extreme value distribution. In the main text, we model the supply side of the economy as a competitive endowment economy. This choice is deliberate and designed to abstract from inefficiencies that might arise from imperfect competition in product markets or spatial spillovers (Fajgelbaum and Gaubert, 2020). We then proceed to characterize several allocations.

First, we characterize the allocation that is usually solved for in random utility, discrete choice models. We call this the *standard allocation*, but our characterization illustrates the following point: that ex-ante identical households (before the taste shocks are realized) ex-post value resources differently, depending upon their choice. For example, someone choosing a high priced commodity / expensive location will have a high marginal utility of consumption, compared with someone choosing a low priced commodity / cheap location. This observation suggests there are ex-ante Pareto improving trades that could be made either through a market arrangement or by a social planner. We emphasize that this is an *incomplete markets* allocation.

Second, we characterize the complete markets allocation and Pareto efficient allocations. We demonstrate the equivalence between the two. This provides a first and second Welfare theorem for discrete choice economies. Using complete markets allows researchers to pinpoint if and how the *standard allocation* departs from the efficient allocation.

Third, we show that every Pareto efficient allocation corresponds to the maxima of a problem in which a planner maximizes a standard social welfare function under some vector of social welfare weights. This is key, since this is the social welfare function used by economists to evaluate welfare gains from various counterfactuals in discrete choice economies (see: Train, 2009; Anderson et al., 1992).[1] Combined, these results allow researchers to understand any social welfare improving policy under incomplete markets in terms of the degree to which it implements a complete markets allocation. In other words, whenever a researcher computes optimal policy in a discrete choice economy using a standard social welfare function, any conjectured policy that goes toward "completing markets" will be welfare improving, even without externalities such as amenity or production spillovers.

What does "completing markets" look like? The complete markets economy allows households to trade a complete set of contingent claims on every possible realization of the taste shocks. These asset trades essentially allow the household to face one unified budget constraint rather than a budget constraint that applies, state by state as in the standard setting. The unification of the household's budget constraint allows the household to equalize the marginal value of its resources across states and the discrete choices induced by those states. Thus, the household is able to set consumption so that marginal rates of substitution (across the discrete commodities) equal marginal rates of transformation or relative prices. So even though households choose only one commodity to consume ex-post, households ex-ante choose consumption plans in the same way as in a model where all goods are consumed simultaneously.

A first novelty of the complete markets allocation is a fundamentally different choice probability across goods. In the standard setting, the rule describing which good to choose is $\max$ over $u_j + \xi_j$. This is ***not*** the case with complete markets, any other Pareto efficient allocation or the social welfare maximizing allocation. The optimal rule under complete markets is $\max$ over $u_j + \xi_j - u_j' c_j$. The additional term $u_j' c_j$ is novel. It picks up the idea that as a household contemplates different choices, it internalizes the private cost of the choice on the consolidated budget constraint. A planner equates private and social costs and hence has the same expression in its choice rule.

A second novelty of the complete markets allocation is the nature of contracts that are actually traded. In equilibrium, only a subset of contingent claims are valued and traded, even though

[1] Often described in words as "the expectation of the max"; see, for example, Busso et al. (2013, p. 902, paraphrased): "Denote the workers' social welfare as $V = E_\xi \left[\max_j \{u_j^i + \xi_j^i\}\right]$, where the expectation is defined over the ξ_j^i terms." We will build up this social welfare function from first principles.

we started with a complete set of contingent claims on every possible taste shock realization. This subset of claims are indexed only by the choice j, not the taste shocks ξ_j^i. We term these contingent claims *Arrow vouchers*, as they pay off when a commodity choice is chosen. One complaint behind complete markets allocations is that the asset structure is often of impractical consequence. Equivalently, the information requirements on the planner are too large (i.e. it needs to know all the taste shocks). Our result shows that market allocations with a simple contract structure, based on observable actions, can achieve outcomes that are seemingly insurmountable.

The second part of the paper presents a formulation of the discrete choice economy that delivers additional insight into the departure of the standard allocation from efficiency. Our formulation of the discrete choice problem to this point is cumbersome, but practical. We show that a more elegant formulation of the problem exists, cast in terms of direct maximization with respect to choice probabilities. Incomplete markets, complete markets and planning problems can each be simply stated in this way and solved using standard tools from calculus. This formulation provides a marginalist interpretation of how a household or planner wants to allocate its resources across discrete choices. For example, it makes clear the additional marginal conditions available to a consumer facing complete markets, relative to those available in incomplete markets.

The third part of the paper applies our results to several canonical economies and the measurement of the welfare effects of price changes.

First, we study the discrete choice economy of Anderson et al. (1987), where aggregate demand for each good is as if it came from a representative CES consumer. We relax the distributional assumptions on the taste shocks and show that Anderson et al. (1987) is a unique, knife-edge case where incomplete and complete market allocations coincide. The key issue is $\log$ utility, not the taste shock distribution. This suggests that their aggregation result is partially due to the efficiency of the allocation, where aggregation is natural.

Second, we study an economy with constant marginal utility and unit demand. This case directly connects with the industrial organization (IO) literature and the demand for differentiated goods in Berry et al. (1995) and Nevo (2000) in particular. Even with unit demand, we show that these allocations are efficient. The key issue is the constant marginal utility assumption.

Third, we extend our approach to a spatial production economy with spillovers that builds on the quantitative spatial literature espoused in Redding and Rossi-Hansberg (2017) and Fajgelbaum and Gaubert (2020) in particular, with location-specific externalities working through production and amenities. Optimal policies have been often discussed in terms of these *two* externalities. We show that optimal spatial allocations take a form similar to the one in Fajgelbaum and Gaubert (2020) but now understand these to be solving *three* reasons why the spatial competitive equilibrium is inefficient: spillovers in production, amenities, and market incompleteness. If a researcher wants to focus on spillovers, our complete markets formulation gives a way to neutralize welfare gains from policy proposals that operate through insurance. This result is valuable for making sure that policies that are discussed as addressing spillovers actually are addressing spillovers and not simply completing markets.

Our final application illustrates the implications of our results for the empirical measurement of the welfare effects of price changes. A standard result is that equivalent variation to a first order is determined by initial expenditure shares and price changes (Deaton, 1989), while elasticities show up only to a second order. We show that in our discrete choice environment under incomplete markets, equivalent variation to a first order depends on expenditure shares, price changes *and* elasticities of demand. Heterogeneity in price sensitivity (as emphasized recently in Auer et al. 2022) is a first-order determinant of welfare in discrete choice models. In contrast, when markets are complete, we show that the "standard" formula is obtained. For applied research, there are two perspectives on these results. First, we have generalized results like those in Deaton (1989) to hold in discrete choice environments, without appeals to functional forms, but under the assumption of complete markets. Second, care should be taken around the interpretation of these formulas since they depend on consumption truly being either (a) "a little bit of all varieties" or (b) discrete with complete markets.

Several related papers connect closely with the questions we address. In many ways, the basic idea behind our work dates back to Rogerson's (1988) work on indivisible labor supply. Rogerson (1988) studies the problem where individual households can only discretely supply their labor; that is they can only work or not. It is often remembered that Rogerson introduces lotteries to convexify the problem, and they lead to optimal allocations and aggregation with aggregate labor supply elasticities differing from micro elasticities. What is not often remembered is that complete markets are introduced to eliminate the risk induced by the lottery. In the discrete choice environment we consider, randomization by the taste shock naturally convexifies the problem, but the risk associated with the shock remains. In this paper, we complete markets and illustrate the properties of allocations, demonstrate the optimality of these allocations, and derive results useful for applied welfare analysis in these settings.

Fajgelbaum and Gaubert (2020) is an important study that characterizes optimal spatial policies with heterogeneous workers and spillovers. Our environment focuses on a separate issue where heterogeneity is induced via taste shocks, and our focus is on the role that market incompleteness plays. We argue that these issues loom over all discrete choice economies, on top of other economically interesting distortions.[2] The difference in focus and our characterization of

[2] As emphasized above, additive taste shocks have provided a foundation for introducing heterogeneity into spatial models like the one in Diamond (2016), and this case connects with the dynamic discrete choice frameworks in Kennan and Walker (2011), Artuç et al. (2010), Caliendo et al. (2019).

first-best allocations contributes to the discussion of optimal spatial policies by providing an alternative motive for market interventions like place-based transfers. We illustrate this last point in Section 6.3 by considering an economy with externalities working through production and amenities as in Fajgelbaum and Gaubert (2020) and then using our complete markets results to distinguish between these different motives.

Ales and Sleet (2022) study questions closely related to the ones in this paper. In a similarly general choice framework, they study the problem of a government that must raise revenue by taxing income associated with each discrete choice and they characterise the optimal tax structure. Our problem is different; in particular, our planner controls everything and we provide decentralization results through ex-ante trades that can be simply conditioned on ones choice / action. With that said, the issues we highlight are motives a government in their setting must confront, i.e., absent complete markets, optimal tax policy is providing some insurance against taste shocks which may attract households to an expensive commodity / location.

In contemporaneous work, Donald et al. (2023a,b) characterize optimal policies in a dynamic spatial environment and how the welfare effects of spatial shocks are shaped by the dispersion in marginal utility under standard assumptions on utility functions and distributions of shocks. The unique contribution of our paper is the decentralization results around complete markets. We can determine precisely the efficient choice rule, contracts traded and do so without assumptions on utility functions or distributional assumptions on shocks. Moreover, our results in Section 5 complement the planning problem they study by providing a foundation for the social welfare function that they use, which we also used in earlier work of our own (Lagakos et al. 2023; Waugh 2023). Our second welfare theorem then provides an equivalence between the planner's allocation characterized in these previous problems and how that allocation can be supported as a competitive equilibrium. Hence, we provide a bottom-up foundation for discussing welfare in these economies.

1. Economic Environment

This section describes the economic environment.

Households. There are a finite set of N types of households, indexed by θ. Within each type, there is a continuum of ex-ante identical households with mass $\mu(\theta)$. Within each type, households' names are indexed by $i \in [0,\ \mu(\theta)]$. Households consume in the economy.

Goods. There are two types of goods. First, there is a single homogeneous, "outside" consumption good c. The price of this outside good is our numeraire and normalized to one. Second, there are finite "differentiated" goods with names $j = \{1,\ 2\ldots, J\}$; we denote quantities q_j. We restrict each household to choose only one type of the differentiated good to consume; this is the discrete choice aspect of the model.

Preferences. There are several components to preferences. First, households receive a taste shock for each differentiated good $j \in J$. Per the restriction that households must make a discrete choice across differentiated goods, only the shock corresponding to the good consumed actually enters utility.

Following the literature, we model the taste shocks as random variables that are independent and identically distributed across agents i in the population of θ-type households. Importantly, this means our results naturally cover the use of discrete choice models in dynamic settings, in which preference shocks are realized each period and expected utility is naturally evaluated before their realization.

Define a realization of taste shocks for household i as $\boldsymbol{\xi}^i = (\xi_1^i, ..., \xi_j^i, ...\xi_J^i)$. Associated with these J random variables are a cumulative density function $G(\boldsymbol{\xi}^i; \theta)$ and a probability density function $g(\boldsymbol{\xi}^i; \theta)$. At this point, we do not make any functional form assumptions on G. With that said, below we will use the canonical type 1 extreme value distribution as an example for illustrative purposes.

Household i of type θ derives the following utility, conditional on choosing good j:

$$u\Big(c, q_j; j, \theta\Big) + \xi_j^i. \tag{1}$$

Utility depends on quantities of consumption of the homogeneous good c and the differentiated good q_j. We assume that assume the utility function is well behaved in these arguments.

Our specification of utility also allows for properties of differentiated good j to matter through channels other than the direct consumption of the differentiated good. The obvious one is the additive taste shock that is specific to household i, discussed above. The second feature is that the utility function is separately indexed by j, representing the idea that the attributes of j may be more or less valuable relative to other j' products. In the case of products, different goods may have different quality or attributes. In the case of location choice, different locations may have different amenities.

Finally, we index utility by the type of the household θ. Different θ types could have different preferences over consumption of the commodities and value attributes of differentiated goods differently. For example, in the case of location, with θ representing family size, single households could have different valuations for, say, cities, than those of nuclear households.

This generality nests representations of canonical functional forms in discrete choice models. One example is the demand for differentiated goods as in Berry et al. (1995) or Nevo (2000). In these applications the outside good is separable, with unit demand for the differentiated good:

$$u\Big(c, q_j; j, \theta\Big) = \alpha_\theta c + \boldsymbol{\beta}(\theta)\boldsymbol{X}_j.$$

Thus, our specification in (1) can entertain ideas emphasized in the demand estimation literature: heterogeneity in own-price elasticities of demand (α_θ) and heterogeneity in consumers' value of product js attributes or quality, $\boldsymbol{X}_j$, via the vector of parameters $\boldsymbol{\beta}(\theta)$.[3]

A simpler example we use below is the following: no outside good, continuous choice in q_j, no quality differences, and common preferences across θ types:

$$u\Big(c, q_j; j, \theta\Big) = u\Big(q_j\Big).$$

When u is $\log$ and $\boldsymbol{\xi}^i$ is distributed according to a type 1 extreme value distribution, we obtain the case of Anderson et al. (1987), who establish equivalence discrete choice demand and demand from a representative CES consumer.

Another useful example is

$$u\Big(c, q_j; j, \theta\Big) = u\Big(c\Big).$$

This is the natural case when j is a location, and thus j effects show up entirely via the budget constraint. This case corresponds to versions of Rosen-Roback models (Rosen, 1979; Roback, 1982) and more recently in Kline and Moretti (2014) or the quantitative spatial literature (Redding and Rossi-Hansberg 2017; Fajgelbaum and Gaubert 2020). Diamond (2016) is an example that merges an IO-like formulation (discussed above) in a spatial setting.

Endowments. Households of type θ are endowed with the commodities discussed above. All households of type θ have the same endowment. In a decentralized economy, where each differentiated good has price p_j, the value of this endowment is

$$W(\theta) = y_o(\theta) + \sum_j p_j y_j(\theta), \tag{2}$$

$y_j(\theta)$ is type-θ endowment of good j and $y_o(\theta)$ is the endowment of the outside good.

2. The Standard (Incomplete Markets) Equilibrium

In this section, we describe the problems of the actors in the economy, the resource constraints, and then define an equilibrium. We call this equilibrium the "Standard (Incomplete Markets) Equilibrium." It is standard in the sense that this is what virtually everyone computes. However, we add the incomplete markets part as a point of contrast to the complete markets equilibrium that we consider in Section 3.

When characterizing the standard equilibrium, we pose the individual household's problem in

[3] Nakamura and Zerom (2010) uses a similar specification in a macro-price setting application.

an ex-ante sense. That is, before taste shocks are realized, the household formulates a plan that maps realizations of $\boldsymbol{\xi}$ into a commodity choice and quantities. And these plans are chosen to maximize ex-ante expected utility.

Our approach differs from the typical one. The typical approach formulates the problem in an ex-post sense. It starts with a realization of shocks and asks what households do. Then, given the ex-post decision rules, the researcher measures welfare by reconstructing average utility in the population. Within θ types, all households are ex-ante identical, so average utility in the population is the same as expected utility of individuals, given some decision rules.

With incomplete markets, these two approaches (ex-post and our ex-ante approach) are equivalent. The decision rules are the same, and maximized ex-ante utility corresponds to utility in the population when utility is maximized ex-post. The reason why we do everything from an ex-ante perspective is that it motivates the existence of ex-ante trades that can be made to increase utility among individuals and in the population.

Another important implication of our approach is that it maps into the wide application of dynamic discrete choice models. In these settings, the shocks are realized each period, and hence ex-ante expected utility is the obvious criteria for individual maximization.

The Household's Problem. Household i of type θ has expected utility $V^i(\theta)$, given by

$$V^i(\theta) = \int_{\boldsymbol{\xi}} \sum_j x^i_j\big(\boldsymbol{\xi},\theta\big)\Big\{u\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] + \xi^i_j\Big\} g\big(\boldsymbol{\xi};\theta\big)\, d\boldsymbol{\xi}. \tag{3}$$

We introduce the following notation: $x^i_j(\boldsymbol{\xi},\theta)$ is an indicator function that maps $\boldsymbol{\xi}$ into a one if j is chosen and zero otherwise, and $c^i(\boldsymbol{\xi},\theta)$ and $q^i_j(\boldsymbol{\xi},\theta)$ are functions that map $\boldsymbol{\xi}$ into the quantities consumed.

The inside summation in (3) says that for a given vector of taste shocks $\boldsymbol{\xi}$, utility is whatever good is consumed, how much of it is consumed, plus the shock associated with the good chosen. The outside integral integrates over all possible realizations of $\boldsymbol{\xi}$ with density $g(\boldsymbol{\xi},\theta)$. This defines expected utility for household i of type θ.

The household chooses quantities and indicator functions for all possible realizations of the taste shocks to maximize (3), subject to the household's budget constraint:

$$\max_{c^i_j(\boldsymbol{\xi},\theta),\; q^i_j(\boldsymbol{\xi},\theta),\; x^i_j(\boldsymbol{\xi},\theta)} \int_{\boldsymbol{\xi}} \sum_j x^i_j\big(\boldsymbol{\xi},\theta\big)\Big\{u\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] + \xi^i_j\Big\} g\big(\boldsymbol{\xi};\theta\big)\, d\boldsymbol{\xi}\;, \tag{4}$$

$$\text{subject to} \qquad \big[\,\lambda^i(\boldsymbol{\xi},\theta)\,\big]: \quad \sum_j x^i_j(\boldsymbol{\xi},\theta)\big[c^i(\boldsymbol{\xi},\theta) + p_j q^i_j(\boldsymbol{\xi},\theta)\big] \leq\; W(\theta), \quad \text{for all } \boldsymbol{\xi}. \tag{5}$$

In (5), $\lambda^i(\boldsymbol{\xi}, \theta)$ are multipliers on household budget constraints, with one budget constraint for every realization of the vector of shocks. As we discuss more below, the key issue in this problem is the budget constraints. The constraint (5) must hold shock by shock, and thus a household's valuation of its resources will vary across shocks, with the relative scarcity of resources reflected in the multipliers $\lambda^i(\boldsymbol{\xi}, \theta)$. Not stated in this problem is the implicit constraint that $x^i_j(\boldsymbol{\xi}, \theta)$ can equal one for only one j.

Resource Constraints. Finally there are the following resource constraints. Product demand can't exceed product supply, requiring for all $j = 1, \ldots, J$ that

$$\sum_\theta \int_0^{\mu(\theta)} y_j(\theta)\, di \geq \sum_\theta \int_0^{\mu(\theta)} \left\langle \int_{\boldsymbol{\xi}} x^i_j(\boldsymbol{\xi}, \theta) q^i_j(\boldsymbol{\xi}, \theta) g(\boldsymbol{\xi}; \theta)\, d\boldsymbol{\xi} \right\rangle di\,. \tag{6}$$

The right-hand side is aggregate demand, which integrates over household types θ, then households within each type i. By the law of large numbers, $\langle \cdot \rangle$ is total demand for good j by households of type θ. The left-hand side is aggregate supply, which is the endowments each i, θ household is endowed with. For the outside good, a similar condition must hold:

$$\sum_\theta \int_0^{\mu(\theta)} y_o(\theta)\, di \geq \sum_\theta \int_0^{\mu(\theta)} \left\langle \int_{\boldsymbol{\xi}} c^i(\boldsymbol{\xi}, \theta) g(\boldsymbol{\xi}, \theta)\, d\boldsymbol{\xi} \right\rangle di. \tag{7}$$

We now formally define the standard (incomplete markets) equilibrium.

Definition 1 (The Standard (Incomplete Markets) Equilibrium.) *An equilibrium consists of allocation for each i and θ type $c^i(\boldsymbol{\xi}, \theta)$, $q^i_j(\boldsymbol{\xi}, \theta)$, $x^i_j(\boldsymbol{\xi}, \theta)$ and prices p_j, such that*

i. allocations ($c^i(\boldsymbol{\xi}, \theta)$, $q^i_j(\boldsymbol{\xi}, \theta)$, and $x^i_j(\boldsymbol{\xi}, \theta)$) satisfy the household's problem in (4);

ii. resource constraints (6), (7) *are satisfied.*

2.1. Properties of the Household's Problem

In this section, we characterize properties of the allocations that satisfy the household problem. Yes, this problem has been solved many times. But we do so in a rather different way that facilitates the rest of the analysis. Appendix A details our approach; below, we state the main results.

First, consumption allocations must satisfy the properties

$$u_c\big[c(j, \theta), q_j(\theta); j, \theta\big] = \lambda_j(\theta) \quad \text{and} \quad u_q\big[c(j, \theta), q_j(\theta); j, \theta\big] = \lambda_j(\theta)\, p_j, \tag{8}$$

where u_c is the marginal utility of outside good consumption and u_q is the marginal utility of the differentiated good. Households equate marginal utility of consumption to marginal costs, measured by the multiplier on the budget constraint times the price.

Notice that we purposely dropped the indexing by the taste shock in several ways. This is a result, not a typo or an attempt to save on notation. First, $c(j,\theta)$ no longer depends upon the taste shock but now inherits dependence upon the choice of the differentiated good j. Similarly, $q_j(\theta)$ does not vary with the taste shock. The multipliers that were indexed by events $\boldsymbol{\xi}$ are now re-indexed by the choice, not the shock. All of these arguments follow from the observation that the taste shock does not affect marginal conditions and the taste shock affects the multiplier only through the choice j, not the shock per se. Finally, these same arguments — along with identical resources and preferences for all $i \in \theta$ — imply the household's identity also does not matter.

The second aspect of the solution is the discrete choice. Optimal $x_j^i(\boldsymbol{\xi},\theta)$ takes the form

$$x_j^i(\boldsymbol{\xi},\theta) = \begin{cases} 1, & \text{if } u\big[c(j,\theta), q_j(\theta); j, \theta\big] + \xi_j^i \;\; \geq \;\; \max\limits_{j'} \Big\{ u\big[c(j',\theta), q_{j'}(\theta); j', \theta\big] + \xi_{j'}^i \Big\} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

This is essentially the starting point from the literature. But we did not start here. As we discuss in Appendix A, we first form the Lagrangian from (4), then make a formal variational argument to obtain (9). Thus, our ex-ante approach delivers the same choice rule that would come from an approach that starts with realized shocks.

We can integrate the function in (9) across the taste shocks and arrive at standard discrete choice results. If the shocks are distributed type 1 extreme value with shape parameter η_θ, the probability a household of type θ chooses choice j is

$$\rho_j(\theta) = \exp\left(\frac{u\big[c(j,\theta), q_j(\theta); j, \theta\big]}{\eta_\theta}\right) \Bigg/ \sum_{j'} \exp\left(\frac{u\big[c(j',\theta), q_{j'}(\theta); j', \theta\big]}{\eta_\theta}\right). \tag{10}$$

This is exactly what one would expect if, for example, one were flipping through Train's (2009) textbook presentation. With a continuum of $i \in \theta$, the law of large numbers applies, and $\rho_j(\theta)$ is also the measure of type-θ households consuming j. Then type θ demand in (6) becomes $\mu(\theta)\rho_j(\theta)q_j(\theta)$.

All of this is natural. The conditions in (8) equate ratios of marginal utility between the outside and differentiated good with relative prices. Condition (9) says that given these optimal consumption plans, do the best possible given the shocks that were realized. Risk associated with the event $\boldsymbol{\xi}$ does not appear to directly affect consumption. These are the arguments one might put forward to argue that nothing is wrong with this economy.

Except for one thing. The conditions in (8) do *not* deliver a "risk sharing"-like condition where the shadow value of resources (the multipliers) is equated *across* events. In allocations where risk is shared (see, e.g., Townsend 1994 or Backus and Smith 1993), there would be a condition like

$$u_q\big[c(j,\theta), q_j(\theta); j, \theta\big]/p_j = \lambda(\theta) \quad \forall\, j. \tag{11}$$

Here, the j index is absent from the multiplier: across all states of nature (the taste shocks directly and the choices they induce) consumption is such that resources are valued equally. This is *not* occurring in the standard allocation when one inspects (8).

The problem in (8) is that the events $\boldsymbol{\xi}$ indirectly show up in the choice through the budget constraint. Resources are more valuable when $\boldsymbol{\xi}$ leads to a choice of a high price good (expensive city, or low paying job). In these states, the multiplier $\lambda_j(\theta)$ will be high. In contrast, in the event that $\boldsymbol{\xi}$ leads to the choice of a low price good (cheap city, or high paying job), the multiplier $\lambda_j(\theta)$ is low, resources are abundant, and marginal utility is low. A better arrangement would have a bit more resources available to the household that likes the high price commodity and a bit less available to the household that likes the low price commodity, up to the point that marginal utility across choices is equated. However, to paraphrase Rogerson (1988, p. 11), *making choices discrete creates a barrier to trade.*

What trading opportunities and market arrangements can achieve a condition like (11)? How is commodity choice affected? How do these market arrangements line up with Pareto efficient allocations? Do Pareto efficient allocations line up with social welfare maximizing allocations? These are the questions that we answer next.

3. Complete Markets Equilibrium and First Welfare Theorem

In this section we allow agents to trade contingent claims that pay out given the realizations of the taste shocks. This is our complete markets equilibrium. We argue this equilibrium is Pareto efficient and then show how it differs from the standard equilibrium. Put simply, if you assess welfare in a discrete choice model, this is the benchmark allocation that maximizes it.

3.1. Insurance Contracts

Households purchase $a^i(\boldsymbol{\xi},\theta)$ many claims at the state price $\varphi(\boldsymbol{\xi},\theta)$. These claims pay out $a^i(\boldsymbol{\xi},\theta)$ upon the realization of the taste shock vector $\boldsymbol{\xi}$ and zero otherwise. There are no restrictions on short-selling: $a^i(\boldsymbol{\xi},\theta) \in (-\infty,\infty)$.

Competitive firms provide these insurance contracts to households. Insurance firms determine the state prices; otherwise, they are simply veils. These firms operate by selling insurance contracts $a^i(\boldsymbol{\xi}, \theta)$ minus the cost. The cost is the probability that event $\boldsymbol{\xi}$ occurs times the quantity of insurance provided. The PDF $g(\boldsymbol{\xi};\theta)$ determines the probability of the event $\boldsymbol{\xi}$ occurring. Competitive pricing in the market for each contract yields zero profits and implies that the household faces actuarially fair prices:

$$\varphi(\boldsymbol{\xi}, \theta) = g(\boldsymbol{\xi};\theta). \tag{12}$$

At this point, we emphasize how rich this contract structure is. Ex-ante, households buy and sell claims against every possible realization. Ex-post, paying claims requires identifying households $\boldsymbol{\xi}$-by-$\boldsymbol{\xi}$. This seems implausibly rich and of no practical purpose, so much so that you might stop reading. Don't. As we show below, the complete markets equilibrium can be supposed by a much simpler, more practical market structure.

3.2. The Household Problem with Complete Markets

The household's problem with complete markets is

$$\max_{a^i(\boldsymbol{\xi},\theta),\ c^i(\boldsymbol{\xi},\theta),\ q^i_j(\boldsymbol{\xi},\theta),\ x^i_j(\boldsymbol{\xi},\theta)} \int_{\boldsymbol{\xi}} \sum_j x^i_j(\boldsymbol{\xi},\theta)\Big\{u\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] + \xi^i_j\Big\} g\big(\boldsymbol{\xi};\theta\big)\, d\boldsymbol{\xi}, \tag{13}$$

$$\text{subject to} \quad \sum_j x^i_j(\boldsymbol{\xi},\theta)\big[c^i(\boldsymbol{\xi},\theta) + p_j q^i_j(\boldsymbol{\xi},\theta)\big] \le\ W(\theta) + a^i(\boldsymbol{\xi},\theta) \quad \forall\, \boldsymbol{\xi}, \tag{14}$$

$$\int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta) a^i(\boldsymbol{\xi},\theta) d\boldsymbol{\xi} = 0. \tag{15}$$

A household chooses contingent claims, consumption quantities, and commodity choices for every possible state, subject to two constraints. The first constraint is that consumption equals labor income and amount of insurance purchased, state by state. The second constraint says the household's net asset position must be zero.

To illustrate the key property of this problem, substitute all budget constraints (14) into the constraint for assets (15). The resulting problem is

$$\max_{a^i(\boldsymbol{\xi},\theta),\ c^i(\boldsymbol{\xi},\theta),\ q^i_j(\boldsymbol{\xi},\theta),\ x^i_j(\boldsymbol{\xi},\theta)} \int_{\boldsymbol{\xi}} \sum_j x^i_j(\boldsymbol{\xi},\theta)\Big\{u\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] + \xi^i_j\Big\} g\big(\boldsymbol{\xi};\theta\big)\, d\boldsymbol{\xi}, \tag{16}$$

$$\text{subject to} \quad [\ \lambda^i(\theta)\ ]: \quad \int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta)\Big\{W(\theta) - \sum_j x^i_j(\boldsymbol{\xi},\theta)\big[c^i(\boldsymbol{\xi},\theta) + p_j q^i_j(\boldsymbol{\xi},\theta)\big]\Big\} d\boldsymbol{\xi} = 0. \tag{17}$$

The constraint (17) is *the* distinguishing feature between the complete markets problem and the problem in (4). Here (17), allows the household to consolidate all possible outcomes of $\boldsymbol{\xi}$ before their realization.[4] In contrast, in the standard equilibrium, households face different constraints for each outcome $\boldsymbol{\xi}$ and are unable to ex-ante trade across these different outcomes.

Definition 2 defines the complete markets equilibrium.

Definition 2 (The Complete Markets Equilibrium.) *A complete markets equilibrium are allocations $c^i(\boldsymbol{\xi}, \theta)$, $q^i_j(\boldsymbol{\xi}, \theta)$, $a^i(\boldsymbol{\xi}, \theta)$, $x^i_j(\boldsymbol{\xi}, \theta)$ for each i and type θ; goods prices p_j and state prices $\varphi(\boldsymbol{\xi}, \theta)$ such that*

i he consumption allocations ($c^i(\boldsymbol{\xi}, \theta)$, $q^i_j(\boldsymbol{\xi}, \theta)$, and $x^i_j(\boldsymbol{\xi}, \theta)$) and asset positions $a^i(\boldsymbol{\xi}, \theta)$ satisfy the household's problem in (16);

ii state prices satisfy (12);

iii goods and asset markets clear.

3.3. The Complete Markets Equilibrium Is Pareto Efficient

In this section, we establish the first fundamental theorem of welfare economics in this environment, which is that the complete markets equilibrium is a Pareto efficient allocation. Appendix B details the entire argument; below, we sketch out our argument.

To establish this argument, first note that any alternative allocation { $\widetilde{c}^i(\boldsymbol{\xi}, \theta)$, $\widetilde{q}^i_j(\boldsymbol{\xi}, \theta)$, $\widetilde{a}^i(\boldsymbol{\xi}, \theta)$, $\widetilde{x}^i_j(\boldsymbol{\xi}, \theta)$} that is preferred by agent i (in the sense that it provides higher utility than (16)) must imply that

$$\int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi}, \theta) \Big\{ W(\theta) - \sum_j \widetilde{x}^i_j(\boldsymbol{\xi}, \theta) \big[\widetilde{c}^i(\boldsymbol{\xi}, \theta) + p_j \widetilde{q}^i_j(\boldsymbol{\xi}, \theta) \big] \Big\} d\boldsymbol{\xi} < 0. \tag{18}$$

In words, this alternative and preferred allocation is not budget feasible.

Armed with this observation, we argue that the complete markets allocation is a Pareto efficient allocation. Consider an alternative allocation $\widetilde{c}^i(\boldsymbol{\xi}, \theta)$, $\widetilde{q}^i_j(\boldsymbol{\xi}, \theta)$, $\widetilde{a}^i(\boldsymbol{\xi}, \theta)$, $\widetilde{x}_j i(\boldsymbol{\xi}, \theta)$ for all i and θ types that is (i) feasible and (ii) preferred by all i and θ types, and strictly preferred by at least one i. Feasibility implies market clearing conditions hold for each good. Multiplying each

[4] As an analog to dynamic models, (17) can be interpreted as a household's "lifetime" budget constraint, and thus the ability to trade insurance allows all these different outcomes to be combined in a "date 0" way.

through by prices and summing across goods, we get

$$\sum_{\theta}\int_0^{\mu(\theta)} y_o(\theta)di + \sum_j\sum_{\theta}\int_0^{\mu(\theta)} p_j y_j(\theta)di = \tag{19}$$

$$\sum_{\theta}\int_{\boldsymbol{\xi}}\int_0^{\mu(\theta)} \widetilde{c}^i(\boldsymbol{\xi},\theta)g(\boldsymbol{\xi},\theta)\,di\,d\boldsymbol{\xi} + \sum_j\sum_{\theta}\int_{\boldsymbol{\xi}}\int_0^{\mu(\theta)} \widetilde{x}^i_j(\boldsymbol{\xi},\theta)p_j\widetilde{q}^i_j(\boldsymbol{\xi},\theta)g(\boldsymbol{\xi};\theta)\,di\,d\boldsymbol{\xi}. \tag{20}$$

On the left-hand side, i-by-i, these are the households' endowments. On the right-hand side, we can use the equilibrium condition that assets trade at actuarially fair prices (thus swapping out the g density for the φ state price), which gives

$$\sum_{\theta}\int_0^{\mu(\theta)} W(\theta)di = \sum_{\theta}\int_0^{\mu(\theta)}\int_{\boldsymbol{\xi}}\varphi(\boldsymbol{\xi},\theta)\left[\sum_j \widetilde{x}^i_j(\boldsymbol{\xi},\theta)\left[\widetilde{c}^i(\boldsymbol{\xi},\theta)+p_j\widetilde{q}^i_j(\boldsymbol{\xi},\theta)\right]\right]d\boldsymbol{\xi}\,di. \tag{21}$$

The right-hand side is now the sum of households' budget constraints. The fact that the *tilde* allocation is preferred implies that one of these constraints does not hold, and hence

$$\sum_{\theta}\int_0^{\mu(\theta)} W(\theta)di = \sum_{\theta}\int_0^{\mu(\theta)}\int_{\boldsymbol{\xi}}\varphi(\boldsymbol{\xi},\theta)\left[\sum_j x'^i_j(\boldsymbol{\xi},\theta)\left[\widetilde{c}^i(\boldsymbol{\xi},\theta)+p_j q'^i_j(\boldsymbol{\xi},\theta)\right]\right]d\boldsymbol{\xi}di \tag{22}$$

$$> \sum_{\theta}\int_0^{\mu(\theta)} W(\theta)di, \tag{23}$$

where the final inequality follows from (18). This is a contradiction. Thus, an allocation that satisfies the definition of a complete markets equilibrium is a Pareto efficient allocation. Proposition 1 states the result.

Proposition 1 (The First Welfare Theorem.) *The complete markets equilibrium is a Pareto efficient allocation.*

The argument we applied was standard, relying on the idea that there can't be a feasible allocation that is preferred. There was no appeal to first order conditions and the like. The next step is to characterize the optimality conditions associated with complete markets. This provides a clear statement regarding the specific failures of the standard equilibrium and a way to establish a connection between all Pareto efficient allocations and the complete markets equilibrium.

3.4. Properties of the Household's Problem with Complete Markets

In this section, we characterize properties of the allocations that satisfy the household problem with complete markets. Appendix B details our approach; below, we state the main results.

Consumption of the outside and differentiated good must satisfy these first order conditions:

$$u_c\big[c(j,\theta), q_j(\theta); j, \theta\big] = \lambda(\theta) \quad \text{and} \quad u_q\big[c(j,\theta), q_j(\theta); j, \theta\big] = \lambda(\theta) p_j. \tag{24}$$

Again, re-indexing is a result, not a shortcut. Consumption $c^i(\boldsymbol{\xi}, \theta)$, $q^i_j(\boldsymbol{\xi}, \theta)$ depends on θ-dependent preferences and endowments (we can drop i) and the good chosen under $\boldsymbol{\xi^i}$, but not the taste shock itself.

The most important thing to notice that the multiplier $\lambda(\theta)$ does not depend upon the realization of the taste shock nor the choice. With complete markets there is one constraint (not shock-specific constraints as in the standard setting), as the household is now able to shift resources across states of nature.

Because these first order conditions are equated with the same multiplier, we now have a standard risk sharing-type result. Compare a θ-type household consuming j with one consuming alternative good j':

$$\frac{u_c\big[c(j,\theta), q_j(\theta); j, \theta\big]}{u_c\big[c(j',\theta), q_{j'}(\theta); j', \theta\big]} = 1 \quad \text{and} \quad \frac{u_q\big[c(j,\theta), q_j(\theta); j, \theta\big]}{u_q\big[c(j',\theta), q_{j'}(\theta); j', \theta\big]} = \frac{p_j}{p_{j'}}. \tag{25}$$

The ratios of marginal utility across goods should equal their relative price. For the homogeneous outside good, marginal utility is the same *independent* of the identity and price of the j good chosen. For the differentiated good, marginal utility differs only to the extent that relative prices are different.

The conditions in (25) also imply that households consume *as if* the consumption choice of the differentiated good were continuous. That is, marginal rates of substitution across the discrete commodities equal relative prices. Efficiency will imply that these are equated to marginal rates of transformation. Even though households choose only one commodity to consume, households, ex-ante choose consumption plans in the same way that would arise in a model where all goods are consumed simultaneously.

When households can trade all these contingent claims, which good do they actually consume?

Interestingly, the commodity choice rule takes a unique form with

$$x_j^i(\boldsymbol{\xi},\theta) = \begin{cases} 1, & \text{if } u\big[c(j,\theta),q_j(\theta);j,\theta\big] + \xi_j^i - \lambda(\theta)\big[c(j,\theta)+p_jq_j(\theta)\big] \geq \\ & \quad \max_{j'}\Big\{ u\big[c(j',\theta),q_{j'}(\theta);j',\theta\big] + \xi_{j'}^i - \lambda(\theta)\big[c(j',\theta)+p_{j'}q_{j'}(\theta)\big] \Big\} \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

The key novelty is the $\lambda(\theta)\big[\cdot\big]$ term, which is *not* in the rule characterizing the incomplete markets allocation (equation 9). With incomplete markets, the effects of choosing j are siloed into the good-j budget constraint. With complete markets, choosing j imposes a cost across all other states of nature, by removing resources from the aggregated budget constraint (17). This cost is the shadow value of resources $\lambda(\theta)$, times the additional burden $c(j,\theta)+p_jq_j(\theta)$. Efficient choice probabilities now account for this cost.

What are the claims that are actually traded? Here we find that a only a subset of contingent claims are traded in equilibrium. Let $\Xi_j(\theta)$ be the set of all $\boldsymbol{\xi}$ for which a household of type θ chooses j. If $\boldsymbol{\xi} \in \Xi_j(\theta)$, then the household's asset position is

$$a^i(\boldsymbol{\xi},\theta) = W(\theta) - \big[c^i(\boldsymbol{\xi},\theta) + p_jq_j^i(\boldsymbol{\xi},\theta)\big] = W(\theta) - \big[c(j,\theta)+p_jq_j(\theta)\big]. \tag{27}$$

Since the right-hand side is independent of $\boldsymbol{\xi}$ directly, then so is $a^i(\boldsymbol{\xi},\theta)$, which we denote $a(j,\theta)$.

This tells us that the only claims that are actually traded are those that depend upon the choice j, not the taste shocks per se. In other words, the same allocation would have been obtained with a set of securities that paid off conditional on choices. We term these contingent claims **"Arrow vouchers,"** as they pay off in states of nature that coincide with the choice of a commodity — hence our voucher terminology.

We find this result striking. An often-heard criticism of the market structures that we considered is that they are complex and unrealistic, and thus of impractical consequence. These claims may be true. But now, ex-ante, households of a type θ trade only claims to J securities among themselves. Ex-post, payment of claims requires only knowing the good chosen, which is observable. This seems less implausibly rich and of practical purpose. Take, for example, a simple policy of a subsidy to individuals who choose to live in San Francisco. The complete markets allocation says that a component of this is welfare improving, as it provides insurance against waking up with a high preference for living in an expensive city.

A second important observation is that these trades really are about insurance, not redistribution. Observe that trade could restricted to be *within* household types (or just assume that there was only one θ-type in the economy). In this case, identical households will want to trade against the potential ex-post different choices that the taste shocks induce.

Circling back to the choice rule in (26), we can connect the Arrow vouchers with the choices in the following way:

$$x_j^i(\boldsymbol{\xi}, \theta) = \begin{cases} 1, & \text{if } u\big[c(j,\theta), q_j(\theta); j, \theta\big] + \xi_j^i - \lambda(\theta) a(j,\theta) \geq \\ & \qquad \max_{j'} \Big\{ u\big[c(j',\theta), q_{j'}(\theta); j', \theta\big] + \xi_{j'}^i - \lambda(\theta) a(j',\theta) \Big\} \\ \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

The extra term is the asset position required to optimally smooth consumption across choices, evaluated at its opportunity cost, which is the $\lambda(\theta)$ term or the marginal utility of consumption from (25).

If we impose the type 1 extreme value distributional assumption and integrate (28), we obtain the *efficient choice probabilities*:

$$\exp\left(\frac{u\big[c(j,\theta), q_j(\theta), j, \theta\big] - \lambda(\theta) a(j,\theta)}{\eta_\theta}\right) \Big/ \sum_{j'} \exp\left(\frac{u\big[c(j',\theta), q_{j'}(\theta), j', \theta\big] - \lambda(\theta) a(j',\theta)}{\eta_\theta}\right). \tag{29}$$

Equation (29) has two meanings. One is the probability that a θ type agent chooses j. With complete markets, the choice probabilities are also the state price for the Arrow voucher associated with choice j. In other words, the choice probabilities are also the price of purchasing insurance against events that lead to j being chosen.

At this point we have demonstrated several things. First, the complete markets allocation leads to a consolidated budget constraint across states of nature, and thus optimal allocations have a risk sharing condition with marginal utility *across* goods equaling their relative price. Second, complete markets influence choice probabilities with the household now internalizing the cost of one choice versus others across states of nature. Finally, the set of contingent claims needed to implement complete markets needs to span only the choices, not all taste shocks. Proposition 2 summarizes these results.

Proposition 2 (Complete Markets Allocations) *The following conditions characterize the complete markets allocations:*

1. *Consumption allocations must satisfy*

$$u_c\big[c(\theta,j), q_j(\theta); j, \theta\big] = \lambda(\theta) \quad \textit{and} \quad u_{q_j}\big[c(\theta,j), q_j(\theta); j, \theta\big] = \lambda(\theta) p_j.$$

2. *The commodity choice rule is*

$$x_j^i(\boldsymbol{\xi},\theta) = \begin{cases} 1, & \textit{if } u\big[c(j,\theta), q_j(\theta); j, \theta\big] + \xi_j^i - \lambda(\theta) a(j,\theta) \;\; \geq \\ & \qquad \max_{j'} \Big\{ u\big[c(j',\theta), q_{j'}(\theta); j', \theta\big] + \xi_{j'}^i - \lambda(\theta) a(j',\theta) \Big\} \\ & \\ 0, & \textit{otherwise.} \end{cases}$$

3. *"Arrow Vouchers." Asset positions are given by*

$$a(j,\theta) = W(\theta) - \big[c(j,\theta) + p_j q_j(\theta)\big],$$

which are contingent only on the choice j, not the taste shock $\boldsymbol{\xi}$. The state prices for the Arrow Vouchers are the choice probabilities, which with the type 1 extreme value assumption are

$$\exp\left(\frac{u\big[c(j,\theta), q_j(\theta), j, \theta\big] - \lambda(\theta) a(j,\theta)}{\eta_\theta}\right) \Bigg/ \sum_{j'} \exp\left(\frac{u\big[c(j',\theta), q_{j'}(\theta), j', \theta\big] - \lambda(\theta) a(j',\theta)}{\eta_\theta}\right).$$

4. Pareto Efficient Allocations and Second Welfare Theorem

We have shown that the complete markets equilibrium is ***a*** Pareto efficient allocation. We now characterize ***all*** Pareto efficient allocations. This provides the foundation for the second welfare theorem: any Pareto efficient allocation can be decentralized as a complete markets equilibrium allocation with the appropriate ex-ante transfers. Appendix C provides all derivations and details.

We set up the Pareto problem as follows. Fix one household with name (i,θ). The planner chooses allocations of consumption and commodity choice rules to maximize (i,θ) utility subject to (i) resource constraints, and (ii) making all other (k,θ') households no worse than some given level. Below are the details.

The objective function of the planner is to maximize expected utility for (i,θ) household:

$$\max V^i(\theta) = \int_{\boldsymbol{\xi}} \sum_j x_j^i(\boldsymbol{\xi},\theta) \Big\{ u\big[c^i(\boldsymbol{\xi},\theta), q_j^i(\boldsymbol{\xi},\theta); j, \theta\big] + \xi_j^i \Big\} g(\boldsymbol{\xi};\theta) d\boldsymbol{\xi}, \tag{30}$$

where the choice variables are the household's consumption and commodity choices and consumption and commodity choices for *all* other agents in the economy which are indexed by

(k,θ'). The first constraints are the resource constraints:

$$[\,\Lambda_o\,]:\quad \sum_{\theta'}\int_0^{\mu(\theta')} y_o^k(\theta')dk \geq \sum_{\theta'}\int_\xi\int_k x_j^k(\boldsymbol{\xi},\theta')c_j^k(\boldsymbol{\xi},\theta')\;dk\;g(\boldsymbol{\xi},\theta')d\boldsymbol{\xi}, \tag{31}$$

$$[\,\Lambda_j\,]:\quad \sum_{\theta'}\int_0^{\mu(\theta')} y_j^k(\theta')dk \geq \sum_{\theta'}\int_\xi\int_k x_j^k(\boldsymbol{\xi},\theta')q_j^k(\boldsymbol{\xi},\theta')\;dk\;g(\boldsymbol{\xi},\theta')d\boldsymbol{\xi}\quad \forall j, \tag{32}$$

where the new multipliers Λ_j and Λ_o capture the shadow value of each good. The next constraint is the Pareto constraint:

$$[\,\Upsilon^k(\theta')\,]:\;\; V^k(\theta') \leq \int_{\boldsymbol{\xi}}\sum_j x_j^k(\boldsymbol{\xi},\theta')\Big\{u\big[c^k(\boldsymbol{\xi},\theta'),q_j^k(\boldsymbol{\xi},\theta');j,\theta'\big]+\xi_j^k\Big\}g(\boldsymbol{\xi},\theta')d\boldsymbol{\xi}\quad \forall k,\theta'\neq i,\theta. \tag{33}$$

Any allocation must deliver utility level $V^k(\theta')$ or better for every (k,θ') household. Associated with this constraint is the Lagrange multiplier $\Upsilon^k(\theta')$. Any Pareto efficient allocation satisfies this problem.

4.1. Properties of Pareto Efficient Allocations

The characterization of the Pareto problem takes the following form. For household (i,θ), the marginal conditions are as follows (where, again, the notation is a result):

$$u_c\big[c^i(j,\theta),q_j^i(\theta);j,\theta\big]=\Lambda_o\quad\text{and}\quad u_{q_j}\big[c^i(j,\theta),q_j^i(\theta);j,\theta\big]=\Lambda_j. \tag{34}$$

Then for all other (k,θ') households, they are

$$\Upsilon^k(\theta')\,u_c\big[c^k(j,\theta'),q_j^k(\theta');j,\theta'\big]=\Lambda_o\quad\text{and}\quad \Upsilon^k(\theta')\,u_{q_j}\big[c^k(j,\theta'),q_j^k(\theta');j,\theta\big]=\Lambda_j. \tag{35}$$

These conditions mimic what is occurring in the complete markets equilibrium. Adjusted, marginal utility equals the shadow cost of the commodity, which is the multipliers on the resource constraint.

The multipliers $\Upsilon^k(\theta')$ on the Pareto constraints reflect the social value of each individual to the planner. A high social value individual with a large $V^k(\theta')$ will have a large multiplier $\Upsilon^k(\theta')$. Why? The multiplier measures $\partial V^i(\theta)/\partial V^k(\theta')$. Consider a low value and high value (k,θ'). With concave utility, reducing the low value individual's value is achieved by a small decrease in consumption, which slightly increases the goods and utility available to (i,θ); hence, $\partial V^i(\theta)/\partial V^k(\theta')$ is small. Reducing the high value individual's value by the same amount requires a large reduction in consumption, which substantially increases the goods and hence utility available to (i, θ).

The next step is the optimal choice rule describing which commodity is be consumed. The choice rule is given by

$$x_j^k(\boldsymbol{\xi}, \theta') = \begin{cases} 1, & \text{if } u\big[c^k(j,\theta'), q_j^k(\theta'); j, \theta'\big] + \xi_j^k - \frac{1}{\Upsilon^k(\theta')}\Big[\Lambda_o c^k(j,\theta') + \Lambda_j q_j^k(\theta')\Big] \geq \\ & \quad \max_{j'} \Big\{ u\big[c^k(j',\theta'), q_{j'}^k(\theta'); j', \theta'\big] + \xi_{j'}^k - \frac{1}{\Upsilon^k(\theta')}\Big[\Lambda_o c^k(j',\theta') + \Lambda_{j'} q_{j'}^k(\theta')\Big]\Big\} \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

This holds for all (k, θ') households and for (i, θ) by setting the Υ term to one.

Again, this expression looks similar to the complete markets equilibrium. Consider the last additional term $\frac{1}{\Upsilon^k(\theta')}[\cdot]$. Rather than the private cost of insurance into choice j, it is the *social cost* of providing utility to the (k, θ') household. This is the shadow value of outside and differentiated good consumption, evaluated at the shadow value of those resources, Λ_o and Λ_j. This is adjusted by how socially valuable household (k, θ') is.

Proposition 3 summarizes.

Proposition 3 (Pareto Efficient Allocations) *Given utility levels $V^k(\theta')$ for all $k, \theta' \neq i, \theta$, a* ***Pareto efficient allocation*** *is consumption allocations and commodity choice rules $c_j^i(\boldsymbol{\xi}, \theta)$, $q_j^i(\boldsymbol{\xi}, \theta)$, $x_j^i(\boldsymbol{\xi}, \theta)$ for household (i, θ) and all other (k, θ') households, that solve the problem (30) subject to resource constraints (31, 32) and the Pareto constraint in (33).*

The following conditions characterize Pareto efficient allocations:

1. *For agent (i, θ), consumption allocations must satisfy*

$$u_c\big[c^i(j,\theta), q_j^i(\theta); j, \theta\big] = \Lambda_o \quad \textit{and} \quad u_{q_j}\big[c^i(j,\theta), q_j^i(\theta); j, \theta\big] = \Lambda_j.$$

2. *For agent (k', θ'), consumption allocations must satisfy*

$$\Upsilon^k(\theta')\, u_c\big[c^k(j,\theta'), q_j^k(\theta'); j, \theta'\big] = \Lambda_o \quad \textit{and} \quad \Upsilon^k(\theta')\, u_{q_j}\big[c^k(j,\theta'), q_j^k(\theta'); j, \theta'\big] = \Lambda_j.$$

3. *The commodity choice rule is*

$$x_j^k(\boldsymbol{\xi},\theta') = \begin{cases} 1, & \text{if } u\Big[c^k(j,\theta'), q_j^k(\theta'); j, \theta'\Big] + \xi_j^k - \frac{1}{\Upsilon^k(\theta')}\Big[\Lambda_o c^k(j,\theta') + \Lambda_j q_j^k(\theta')\Big] \geq \\ & \quad \max_{j'} \Big\{ u\Big[c^k(j',\theta'), q_{j'}^k(\theta'); j', \theta'\Big] + \xi_{j'}^k - \frac{1}{\Upsilon^k(\theta')}\Big[\Lambda_o c^k(j',\theta') + \Lambda_{j'} q_{j'}^k(\theta')\Big]\Big\} \\ 0, & \text{otherwise,} \end{cases}$$

which holds for all (k,θ') households. For the (i,θ) household, set the Υ term to one.

4.2. Second Welfare Theorem

We now have a basis for the Second Welfare Theorem.

Proposition 4 (The Second Welfare Theorem.) *Any Pareto efficient allocations can be decentralized as a complete markets equilibrium.*

This is immediate from comparing the content of Proposition 2 with complete markets to Proposition 3. Consumption allocations and choice rules are identical if $\Lambda_0/\Upsilon^k(\theta') = \lambda(\theta')$ and $\Lambda_j/\Upsilon^k(\theta') = \lambda(\theta')p_j$. To align the complete markets allocation to the Pareto efficient allocation, then one simply needs to find the appropriate ex-ante reallocation of resources to attain the correct multipliers on complete markets budget constraints, $\lambda(\theta')$.

4.3. Social Welfare Maximization

We have worked exclusively with the Pareto problem, but it should not be surprising that a planner maximizing a straightforward social welfare function attains the same allocation. Proposition 5 summarizes this finding. Appendix D provides all derivations and details.

Proposition 5 (Social Welfare Maximizing Allocations) *Let $\psi^i(\theta)$ be a vector of social welfare weights. Define the **Social Welfare Function** as*

$$\mathcal{W}_\psi = \sum_\theta \int_i \psi^i(\theta) \int_{\boldsymbol{\xi}} \sum_j x_j^i(\boldsymbol{\xi},\theta) \Big\{ u\Big[c^i(\boldsymbol{\xi},\theta), q_j^i(\boldsymbol{\xi},\theta); j, \theta\Big] + \xi_j^i \Big\} g(\boldsymbol{\xi},\theta) d\boldsymbol{\xi}\, di. \tag{37}$$

*Then a **social welfare maximizing allocation** is consumption allocations and commodity choice rules and $c_j^i(\boldsymbol{\xi},\theta)$, $q_j^i(\boldsymbol{\xi},\theta)$, $x_j^i(\boldsymbol{\xi},\theta)$ for all i,θ to maximize $\mathcal{W}_\psi$ subject to resource constraints. The following conditions characterize the allocation:*

1. *For all i, θ, consumption allocations must satisfy:*

$$\psi^i(\theta)\; u_c\big[c^i(\theta), q^i_j(\theta); j, \theta\big] = \Lambda_o \quad \textit{and} \quad \psi^i(\theta)\; u_{q_j}\big[c^i(\theta), q^i_j(\theta); j, \theta\big] = \Lambda_j. \tag{38}$$

2. *For all i, θ the commodity choice rule is*

$$x^i_j(\boldsymbol{\xi}, \theta) = \begin{cases} 1, & \textit{if } \; u\big[c^i(j,\theta), q^i_j(\theta); j, \theta\big] + \xi^i_j - \frac{1}{\psi^i(\theta)}\Big[\Lambda_o c^i(j,\theta) + \Lambda_j q^i_j(\theta)\Big] \;\; \geq \\ & \quad \max\limits_{j'} \Big\{ u\big[c^i(j',\theta), q^i_{j'}(\theta); j', \theta\big] + \xi^i_{j'} - \frac{1}{\psi^i(\theta)}\Big[\Lambda_o c^i(j',\theta) + \Lambda_{j'} q^i_{j'}(\theta)\Big]\Big\} \\ & \\ 0, & \textit{otherwise.} \end{cases} \tag{39}$$

This allocation is a Pareto efficient allocation and coincides with a complete markets allocation under some ex-ante transfers.

The last statement in Proposition 5 follows from a comparison with the results in Proposition 3. There is a clear mapping from $\psi^i(\theta)$ to the multipliers on the utility constraints in the Pareto problem. And a similar comparison between the results in Proposition 2 and the conditions from the results above lead to the conclusion that one needs only to find the correct transfers to align the complete markets allocation and the one that solves the social planning problem above.

Taking stock, we have shown that the objective function that researchers typically use to evaluate welfare across counterfactual policy experiments in discrete choice economies, is maximized in a decentralized competitive equilibrium with complete markets, and that the associated allocations are Pareto efficient. This establishes that in an incomplete markets economy, policies that act in the direction toward completing markets will tend to be welfare improving.

5. An Alternative Formulation of the Individuals Objective Function

This section provides a representation of the objective function in (37) under the type 1 extreme value assumption that (i) is easy to solve using standard calculus and (ii) provides a marginalist interpretation as to how a household or planner wants to allocate its resources across discrete choices. Under this distributional assumption, we can cast the objective function and budget / resource constraints directly in terms of choice probabilities, which are then directly chosen, rather than locating the discrete choice rules that we have previously looked for, followed by tedious integration against $G(\boldsymbol{\xi})$.

Appendix E details the entire argument and works through a more general example; below, we sketch out our argument.

We build on three observations. First, in each problem, we started with the household i of type θ having expected utility given by

$$V^i(\theta) = \int_{\boldsymbol{\xi}} \sum_j x_j^i\big(\boldsymbol{\xi}, \theta\big)\Big\{u\big[c^i(\boldsymbol{\xi}, \theta), q_j^i(\boldsymbol{\xi}, \theta); j, \theta\big] + \xi_j^i\Big\} g\big(\boldsymbol{\xi}; \theta\big)\, d\boldsymbol{\xi}.$$

This is the objective function that shows up in the household's incomplete markets problem (4); the complete markets problem (13); the Pareto problem (33); and social welfare maximization problem (37).

Second, in each problem, the optimal choice rule $x_j^i(\boldsymbol{\xi}, \theta)$ is always of the form:

$$x_j^i(\boldsymbol{\xi}, \theta) = \begin{cases} 1, & \text{if } \widetilde{V}_j(\theta) + \xi_j^i \ \geq \ \max\limits_{j'} \left\{ \widetilde{V}_{j'}(\theta) + \xi_{j'} \right\} \\ 0, & \text{otherwise.} \end{cases} \tag{40}$$

Across problems, $\widetilde{V}_j(\theta)$ varies. For example, in the incomplete markets problem, it would just be the deterministic part of the households utility function; in the complete markets setting, it includes the multipliers on the households budget constraint.

Third, in each problem, consumption quantities are independent of the taste shock ξ_j, as ξ_j never appears in the first order condition for $q_j(\boldsymbol{\xi}, \theta)$. This is a direct consequence of the additivity of the shocks.

Combined, these observations allow us to re-express the objective function in (37) in the following way:

Proposition 6 (An Equivalent Objective Function) *Assuming the taste shocks are type 1 extreme value distributed, an individual's objective function can be recast as*

$$V^i(\theta) = \sum_j \rho_j^i(\theta)\Big[u\big[c^i(\theta), q_j^i(\theta); j, \theta\big] - \eta_\theta \log \rho_j^i(\theta)\Big], \tag{41}$$

where—in addition to consumption quantities—choice probabilities are control variables of the household or planner.

The first term in (41) is just expected utility over the deterministic part of utility. However, here the household or planner has control over expected utility and must trade off putting more weight on high u choices versus selection induced by the taste shock. This is where the second term in (41) enters—which is positive and decreasing in ρ_j—like a selection correction term:

choosing a higher probability on choice j means incurring taste shocks that are progressively worse, and hence overall utility is penalized.[5]

Given (41), it is straightforward to maximize utility in the incomplete markets problem in (4), the complete markets problem in (13), the Pareto problem in (33), and the social welfare maximization problem (37). In each case we can easily write down the associated budget constraints and resource constraints with an additional constraint that restricts the choice probabilities to be probabilities; that is they must sum to one. For example, with the incomplete markets problem, the shock-by-shock constraints are

$$\underbrace{c(j,\theta) + p_j q_j(\theta) \leq W(\theta),}_{\text{Incomplete markets: Type-}\theta\text{, Good-}j} \qquad \text{for each } j \in \{1, \ldots, J\}.$$

With complete markets, the budget constraint is now

$$\underbrace{\sum_j \rho_j(\theta) \big[c(j,\theta) + p_j q_j(\theta) \big] \leq W(\theta).}_{\text{Complete markets: Type-}\theta\text{, Consolidated}}$$

Note that we have used the fact that in the competitive equilibrium with complete markets, Arrow securities are competitively priced, and hence the state price of the securities is simply the choice probability ρ_j. This illustrates a feature of the problem that the state prices are endogenous objects, not given by nature. As a household contemplates a higher ρ_j, the price of the Arrow securities that pay off conditional on purchasing j increases one for one.[6] The decentralization via Arrow vouchers is even clearer here, where $\rho_j(\theta)$ is the price of a j voucher.

Example: Complete Markets. To further illustrate how this works, consider an example with complete markets. To simplify the presentation, we drop the outside good and the θ types, and the utility function now depends only upon consumption of the good chosen q_j. Our new

[5]The type 1 extreme value assumption is not necessary, per se, for a representation like (41). Alternative distributional assumptions would give rise to different selection correction terms. The special property of the type 1 extreme value distribution is that the selection correction depends only on ρ^i_j, and is constant elasticity with parameter η_θ: $\mathbb{E}[\xi^i_j|\text{Choose } j] = -\eta_\theta \log \rho^i_j$, depends only on ρ^i_j. It is also worth noting how the selection correction with type 1 extreme value distribution takes the form of entropy and in turn connects with the rational inattention foundation for the logit demands in Matejka and McKay (2015).

[6]An analogy to a different literature may help. In problems of consumer borrowing with endogenous default, individuals understand the price schedule $\varphi(b)$ associated with increasing b by borrowing more. In our case this *schedule* is simply $\varphi(\rho_j) = \rho_j$. Prices depend on observable actions.

formulation allows one to take first order conditions in q_j and ρ_j:

$$q_j: \qquad u'(q_j) = \lambda p_j \tag{42}$$

$$\rho_j: \qquad u(q_j) - \eta \log \rho_j = \lambda p_j q_j + \chi + \eta, \tag{43}$$

where χ is the multiplier on the choice probability constraint, $\sum_j \rho_j = 1$.

The second condition presents a marginalist interpretation of optimal choice probabilities, and equates the marginal benefit of a change in ρ_j to its marginal cost. The marginal benefit of a higher ρ_j is the increased likelihood of receiving the utility from consuming j and the associated taste shock, holding consumption and selection fixed. The marginal cost of a higher ρ_j includes (i) tightening the consolidated budget constraint by expenditure $p_j q_j$, (ii) reducing probabilities available for other goods, and (iii) a worsening of the expected taste shock, with elasticity η. Combining these, the choice probability is

$$\rho_j = \exp\left(\frac{\left[u(q_j) - u'(q_j)q_j\right]}{\eta}\right) \Bigg/ \sum_{j'} \exp\left(\frac{\left[u(q_{j'}) - u'(q_{j'})q_{j'}\right]}{\eta}\right).$$

We think this approach is useful and informative. First, the type 1 extreme value assumption is pervasive. Second, the objective function in Proposition 6 has been used in previous work (Lagakos et al., 2023; Waugh, 2023; Donald et al., 2023a), and these arguments provide a foundation for it. Third, this formulation can be used to easily state and solve competitive and planner allocations in complex discrete choice economies, as we demonstrate in Section 6.3.

6. Examples of Specific Economies

In this section, we focus on several important examples of discrete choice economies. Specifically, we study the (i) Anderson et al. (1987) economy giving CES demands, (ii) the linear utility implementation of Berry et al. (1995) popularized by Nevo (2000), and (iii) a general equilibrium spatial economy building on Fajgelbaum and Gaubert (2020). In the first two, we show how functional form assumptions imply knife-edge cases where the incomplete markets allocation is efficient. In the third, we show the spatial model (even without spillovers) is generically inefficient without complete markets.

6.1. Anderson et al. (1987) Economy

One example of interest is the Anderson et al. (1987, henceforth, ADT) economy. In their economy aggregated demands for each commodity are as if they came from a representative consumer with a CES utility function over the commodities. Given what we have learned, one may make the connection that aggregation is usually possible when complete markets are available.

We show that this is indeed the case and that the ADT economy has the unique feature that the incomplete markets allocation aligns with the complete markets allocation, and hence is efficient. This result does not depend on the taste shock distribution.

As a special case of our general discrete choice economy, the ADT economy has the following features: (i) no outside good, (ii) continuous choice in q_j, (iii) no quality differences, (iv) common preferences across θ types, and (iv) different endowments across θ types. Finally, and most importantly, preferences in the ADT economy are $\log$ over the differentiated good, $u(q_j) = \log q_j$. Given these assumptions, we have the following objective function and budget constraints:

$$\int \sum_j x_j^i(\boldsymbol{\xi},\theta)\Big[u(q_j(\boldsymbol{\xi},\theta)) + \xi_j^i\Big]dG(\boldsymbol{\xi}) \ , \qquad \underbrace{p_j q_j(\theta) = W(\theta)}_{\text{Incomplete markets, for each } j} \qquad \underbrace{\int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi}) x_j^i(\boldsymbol{\xi},\theta) p_j q_j(\theta)\, d\boldsymbol{\xi} = W(\theta)}_{\text{Complete markets}} .$$

In this economy, the allocations under incomplete markets and complete markets satisfy the same conditions. Consider the first order condition for $q_j(\boldsymbol{\xi})$ with incomplete markets:

$$\lambda_j(\theta) = \frac{u'(q_j(\theta))}{p_j} = \frac{1}{p_j q_j(\theta)} = \frac{1}{W(\theta)} = \frac{1}{p_{j'} q_{j'}(\theta)} = \frac{u'(q_j(\theta))}{p_j} = \lambda_{j'}(\theta). \tag{44}$$

Simply consuming $q_j(\theta) = W(\theta)/p_j$, as dictated by the budget constraint, also equalizes the marginal value of wealth across goods (as in (25)). With $\log$ preferences, when a household consumes a good that is 1 percent more expensive, marginal utility increases by exactly 1 percent. This implies that the multiplier is constant, regardless of the good that is chosen, delivering the key condition of the complete markets allocation.

The next step is the commodity choice. Starting from the complete markets choice rule (26), we substitute in the first order condition for $q_j(\theta)$: $\lambda(\theta) p_j = u'(q_j(\theta))$. This gives the first expression here:

$$x_j(\boldsymbol{\xi},\theta) = \begin{cases} 1, & \text{if } \ u(q_j(\theta)) + \xi_j^i - u'(q_j(\theta))q_j(\theta) \ \geq \\ & \qquad \max_k \big[\ u(q_k(\theta)) + \xi_k - u'(q_k(\theta))q_k(\theta)\ \big] \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} 1, & \text{if } \ u(q_j(\theta)) + \xi_j^i \ \geq \\ & \qquad \max_k \big[\ u(q_k(\theta)) + \xi_k\ \big] \\ 0, & \text{otherwise.} \end{cases} \tag{45}$$

The second equality is because $u'(q)q = 1$ under $\log$ preferences, which gives the incomplete markets choice rule. Thus, we have established that the ADT economy has the unique feature that the incomplete markets allocation aligns with the complete markets allocation and hence is efficient.

A feature of our argument is that we did not invoke a distributional assumption on the taste shocks. The alignment between complete and incomplete markets in this setting is only about the shape of the utility function. This has implications regarding the aggregation result of Anderson et al. (1992). Clearly, type 1 extreme value assumption is important in delivering CES demand functions. However, the fact that their economy is efficient suggests alternative aggregation results are possible with alternative distributions on the taste shocks.

6.2. Constant Marginal Utility, Unit Demand

The next example is settings with constant marginal utility and unit demand on the differentiated good. This case directly connects with the IO literature and the demand for differentiated goods, as in Berry et al. (1995) and Nevo (2000) in particular. The allocations that obtain turn out to be efficient.

We build on the survey of Nevo (2000), who specifies the u function in (1) as

$$u\Big(c, 1; j, \theta\Big) = \alpha_\theta c + \boldsymbol{\beta}(\theta)\boldsymbol{X}_j.$$

Here, α_θ is the marginal utility of outside good consumption, and the second term models interactions between household type and product characteristics. An important feature of this economy is that even though there is unit demand, the cost of buying one good or another affects consumption of the outside good through the budget constraint: $c(\theta) + p_j = W(\theta)$. This observation implies that the household's first-order condition on the outside good generically depends upon the commodity choice:

$$u_c\Big(W(\theta) - p_j, 1; j, \theta\Big) = \lambda_j(\theta) \quad \Rightarrow \quad \alpha_\theta = \lambda_j(\theta) \quad \Rightarrow \quad \lambda_j(\theta) = \lambda_{j'}(\theta) \;\; \forall j, j'. \tag{46}$$

By definition, linear utility implies that the marginal utility of consumption is equalized across choices, which is exactly the condition that arises with complete markets in (25).

The more interesting question is which commodity should be chosen. The complete markets choice rule is

$$x^i_j(\boldsymbol{\xi}, \theta) = \begin{cases} 1, & \text{if } \; -\alpha_\theta p_j + \boldsymbol{\beta}(\theta)\boldsymbol{X}_j + \xi^i_j \;\; \geq \\ & \qquad \max_{j'} \big[-\alpha_\theta p_{j'} + \boldsymbol{\beta}(\theta)\boldsymbol{X}_{j'} + \xi^i_{j'} \big] \\ 0, & \text{otherwise,} \end{cases} \tag{47}$$

which is the same choice rule that obtains under incomplete markets. Thus, the allocations in this common setting turn out to be efficient.

Note that this equivalence between complete and incomplete markets allocations when demand is discrete does not obtain under $\log$ utility, which is the case studied by Berry et al.

(1995). The use of a linear objective is sometimes discussed as a *first order approximation* to more general utility function as in Berry et al. (1995). Linearizing utility removes issues associated with risk, which restores efficiency, since the gap between incomplete and complete markets allocation is due to the inability to insure across choices.[7] Avoiding risk and insurance issues might be reasonable in applications typically studied by the IO literature. However, the constant marginal utility assumption in some settings, like the purchase of an car as in Berry et al. (1995), might not be reasonable, in which case the issues that we raise regarding market incompleteness and the inefficiency of the 'standard' allocation become salient. Furthermore, as we discuss in Section 7, the welfare impacts of changes in prices (say via changes in market structure) are sensitive to whether the underlying allocation is efficient or not.

6.3. A Spatial Economy

The final economy we consider is a spatial setting that is in the spirit of Rosen (1979), Roback (1982) and the quantitative spatial literature (Redding and Rossi-Hansberg, 2017). In particular, we consider a discrete choice version of the setting with productivity spillovers and amenity spillovers studied in Fajgelbaum and Gaubert (2020).

The setting we describe below is interesting because the spatial economics literature uses discrete location choice models to measure the effects of, and propose policies to address, various externalities and spillovers. But what we have learned so far is that there is an additional market failure, which is market incompleteness. Hence, welfare gains from a particular policy may be due to resolving market incompleteness when a researcher thinks they stem from addressing a spillover. As we show below, our complete markets formulation of the problem provides researchers an apparatus useful for distinguishing how a particular policy improves welfare.

Our spatial environment is the following: each choice j is a location in space. We remove the differentiated good, and we assume the homogeneous good is produced by competitive firms, in all locations, and it is freely traded. The mass of each θ type of household is normalized to one. Households begin "locationless," and the discrete choice nature of the problem is where they should live and work.[8]

Production in each location is

$$Y_j = F_j(\boldsymbol{\rho}_j), \tag{48}$$

where Y_j is output of the homogeneous good in location j, F_j is the production function that may depend upon characteristics of location j, and $\boldsymbol{\rho}_j = (\rho_j(\theta), \rho_j(\theta') \dots)$ is the entire vector of

[7] An obvious parallel is that issues of risk and insurance cannot be studied in first order approximations of macroeconomics models.

[8] An alternative interpretation is to treat our θ types as a household's initial starting point, and thus the choice probability $\rho_j(\theta)$ characterizes the mass of migrants from location θ to location j.

all different type θ households working in location j. This last point means that output in location j may depend on the mix of type θ households working in that location. This production function could also exhibit an external productivity spillover.

Utility is

$$u\big[c^i(\boldsymbol{\xi},\theta)\big] + A_j(\boldsymbol{\rho}_j;\theta) + \xi^i_j, \tag{49}$$

where $A_j(\boldsymbol{\rho}_j;\theta)$ is a j- and θ-specific amenity value. As with production, the dependence of A_j upon $\boldsymbol{\rho}_j$ means that the amenities in j valued by type θ households depend upon the entire mix of households residing in that location. This function is treated as a spillover in the sense that individuals take this function as given and they do not internalize how their choices influence it.

Throughout this discussion, we employ a the type 1 extreme value distribution with shape parameter η_θ for each type. Given this assumption, we build on our streamlined presentation of the discrete choice problem in Section 5 and solve these problems by directly choosing choice probabilities.

The Incomplete Markets Allocation. The choice probability in the incomplete markets allocation, $\rho^{IM}_j(\theta)$, is

$$\rho^{IM}_j(\theta) \propto \exp\left\{\eta_\theta^{-1}\Big(u\big[w_j(\theta)\big] + A_j(\boldsymbol{\rho}_j;\theta)\Big)\right\}. \tag{50}$$

The constant of proportionality is the term in the denominator, which is just the sum across all the j terms. In the incomplete markets allocation, the choice probability is just about utility and the amenity value in that location. Spillovers (production and amenity) are not internalized. And because of incomplete markets, consumption equals the prevailing wage rate $w_j(\theta)$ in location j for type θ households.

The Social Welfare Maximizing Allocation. Using our results from Section 5, we get the following social welfare maximizing planning problem:

$$\max_{c(j,\theta),\rho_j(\theta)} \sum_\theta \sum_j \rho_j(\theta) \Big\{ u\big[c(j,\theta)\big] + A_j(\boldsymbol{\rho}_j;\theta) - \eta_\theta \log \rho_j(\theta) \Big\}, \tag{51}$$

$$\text{subject to} \quad [\,\Lambda\,] \ \sum_j F_j(\boldsymbol{\rho}_j) \ \geq \ \sum_\theta \sum_j \rho_j(\theta) c(j,\theta) \tag{52}$$

$$[\,\Lambda_\theta\,] \ 1 \ = \ \sum_j \rho_j(\theta) \quad \forall \theta, \tag{53}$$

where we simplified the problem with common Pareto weights across individuals and types. The solution to this problem is characterized by two first order conditions. First, there is the standard first order condition for consumption:

$$u_c\big[c(j,\theta)\big] = \Lambda. \tag{54}$$

This equates marginal utility across locations. An important observation is that this condition implies that consumption is equalized across space, and hence $u\big[c(j,\theta)\big]$ is equalized across space.

Second, the first order condition for the choice probability is

$$V_j(\theta) - \eta_\theta + \Lambda \frac{\partial F_j}{\partial \rho_j(\theta)} + \sum_{\theta'} \frac{\partial A_j(\theta')}{\partial \rho_j(\theta)/\rho_j(\theta)} = \Lambda c(j,\theta) + \Lambda_\theta, \tag{55}$$

where the $V_j(\theta)$ term is just the total expected utility level associated with choice j (the term inside the curly brackets of 51). This equation is essentially the same as Fajgelbaum and Gaubert's (2020) equation (22).[9] It equates the social marginal gain of a household in location j to its social marginal cost. The marginal gain reflects (i) the level of utility (net of the change in the marginal taste shock, $-\eta_\theta$), (ii) how much extra output is delivered, and (iii) the marginal improvement in amenities for everybody. The marginal cost is just the value of consumption in that location and the reduction in type θ''s capacity to be allocated to other locations, which is valued at Λ_θ.

From (55) we can solve for the social planner's choice probability, $\rho_j^{SP}(\theta)$, which is:

$$\rho_j^{SP}(\theta) \propto \exp\left\{ \eta_\theta^{-1} \left(A_j(\boldsymbol{\rho}_j;\theta) + u_c \frac{\partial F_j}{\partial \rho_j(\theta)} + \sum_{\theta'} \frac{\partial A_j(\theta')}{\partial \rho_j(\theta)/\rho_j(\theta)} \right) \right\}. \tag{56}$$

[9]The main difference is the first two terms. This shows up only because, with the taste shocks, utility is not necessarily equated across locations, and thus a marginal change in the choice probability shifts the level of the taste shock, which influences the selection on the taste shock. This is summarized by the η_θ term.

Interestingly, because the planner is equalizing marginal utility, the consumption and utility terms don't show up, because they don't vary with location. In the efficient allocation, location choice depends only on location-varying amenities, and the terms that capture the marginal effect of location choice on production and amenities. The interpretation is then that the planner is essentially making amenity-adjusted output as large as possible, then redistributing appropriately.

The Complete Markets Allocation. There are several market failures when considering how the incomplete markets choice probabilities stack up relative to those chosen by the planner. An advantage of the complete markets allocation is that we can isolate any issues regarding incomplete insurance against the taste shocks. In the complete markets allocation, the first order condition for consumption is

$$u_c\big[c(j,\theta)\big] = \lambda(\theta), \tag{57}$$

where $\lambda(\theta)$ is the multiplier on type θ households unified budget constraint. Similar to the planner's problem, this condition implies that (i) consumption is equalized across space and (ii) that $u\big[c(j,\theta)\big]$ is equalized across space. With that said, consumption is not necessarily equalized across θ types, because expected wages may differ by type and location, and hence total resources differ by type.

The first order condition for the choice probability becomes

$$V_j(\theta) - \eta_\theta + \lambda(\theta)w_j(\theta) = \lambda(\theta)c(j,\theta), \tag{58}$$

where $w_j(\theta)$ is the prevailing wage in location j for type θ households. Here, the first order condition requires a balance between the private gain from being in a location and the private cost, but neglects the social gains that a planner internalizes. The private gain is the level of utility net of the change in selection on the taste shock (the first two terms) and the extra earnings gained in location j as reflected by the wage rate. The private cost is the value of consumption in that location.

From (58) we can solve for complete markets choice probability, $\rho_j^{CM}(\theta)$, which is

$$\rho_j^{CM}(\theta) \propto \exp\left\{\eta_\theta^{-1}\Big(A_j(\boldsymbol{\rho}_j;\theta) + u_c(\theta)w_j(\theta)\Big)\right\}. \tag{59}$$

Equation (59) has the same flavor as (56), but there are several key distinctions. First, the effect of internalizing the amenity spillovers is not present. And with productivity spillovers, the wage rate does not align with the social marginal product of labor. With that said, it still has an interpretation similar to that of the planner's solution. An individual household should make private income, adjusted for amenity differences, as large as possible and then use Arrow securities to transfer the income across different location choices. Thus, with complete markets there are motives for spatial transfers independent of externalities in production or amenities, and the planner's solution corrects all three.

No Spillovers, Constant Marginal Product of Labor. Another way to distinguish between market incompleteness and externalities in a welfare maximizing allocation is to turn off the spillovers (both productivity and amenity). Here, we do so by considering linear production technologies with TFP levels Z_j, and hence the marginal product of labor is constant. In this case the optimal choice probability of a social planner facing no spillovers collapses to

$$\rho_j^{SP-NS}(\theta) \propto \exp\left\{\eta_\theta^{-1}\left(u_c\, Z_j\right)\right\}. \tag{60}$$

This says that the planner locates households according to each location's productivity, weighted by the (common) marginal utility of consumption. Then, with the absence of amenity differences, the wage rate is Z_j. Hence, it's easy to see that the complete markets choice probability in (59) aligns with the planner's choice probability. Finally, the comparison between (50) and (60) illustrates that the spatial model (even absent spillovers) is generically inefficient without complete markets.

The complete markets / optimal allocation is intuitive: the efficient allocation makes production, net of tastes, as large as possible and then redistributes output ex-post. Moreover, the direction of the transfers is clear. Transfers go from productive / high-wage locations (San Francisco) to unproductive / low-wage locations (Appalachia). Spatial policy would implement this with place-based transfers, which look very much like Arrow vouchers.

7. The Welfare Effects of Changes in Prices

In this final section, we connect our results regarding complete and incomplete markets allocations with what they imply for measuring the welfare impacts of changes in prices.

We start from standard results in consumer theory regarding the measurement of the welfare effects of change in prices. Deaton (1989) is the seminal reference, with the insight that initial budget shares are the sufficient statistics needed to compute the welfare effects of price changes to a first order. This approach has gained increasing popularity because of both the recent availability of itemized household consumption data and the seemingly assumption-free nature of the exercise. Del Canto et al. (2023), Fagereng et al. (2022), and Borusyak and Jaravel (2021) are recent, important applications of this approach. We extend this approach to our discrete choice economy and show that the Deaton (1989)-like results hold in complete markets but fail with incomplete markets.

To simplify the presentation, we strip back our notation and focus on a simplified economy: there is a single household type, only J goods with prices $\mathbf{p} = (p_1, \ldots, p_J)$, and no outside good. Income y is denominated in some numeraire.

Following the literature, we focus on the equivalent variation welfare metric (EV). To define EV, let $V(\mathbf{p}, y)$ be the indirect utility function from the household's choice problem. Then consider some alternative vector of prices $\mathbf{p}' = \mathbf{p} + d\mathbf{p}$. Equivalent variation is the percent change in income ϕ at the old prices that satisfies

$$V\Big(\mathbf{p} + d\mathbf{p}, y\Big) = V\Big(\mathbf{p}, y + dy\Big), \qquad \phi = \frac{dy}{y}. \tag{61}$$

So, in percent terms, how much extra income must be provided at the old prices to make the household indifferent? As an intermediate step, we arrive at a the following expression by taking a first order approximation of both sides of the first equality around the point where $d\mathbf{p} = \mathbf{0}$ and $dy = 0$. This approximation yields

$$\phi \approx \sum_{j=1}^{J} \frac{V_{p_j}\Big(\mathbf{p}, y\Big)}{V_y\Big(\mathbf{p}, y\Big)} \frac{p_j}{y} \Delta \log p_j. \tag{62}$$

Right now, this expression does not look very standard. However, the power of this is that we can compute $V_{p_j}\Big(\mathbf{p}, y\Big)$ and $V_y\Big(\mathbf{p}, y\Big)$ under different assumptions about the choice problem (discrete or continuous choice) and market structure (complete or incomplete markets). Thus, this general approach illustrates how these different problems and settings relate to each other.

7.1. The Continuous Choice Model

First, we consider the continuous choice model. This problem provides an overview of the standard results like those in Deaton (1989).

Let $u(q_1, \ldots, q_J)$ be the utility function from consuming *all* the J goods in continuous quantities that satisfies standard assumptions. Then, working with the connection between the Lagrangian and the indirect utility function, we can apply the envelope theorem to compute $V_{p_j}\Big(\mathbf{p}, y\Big)$ and $V_y\Big(\mathbf{p}, y\Big)$:

$$V_{p_j}\Big(\mathbf{p}, y\Big) = -\lambda q_j \quad , \quad V_y\Big(\mathbf{p}, y\Big) = -\lambda, \tag{63}$$

where λ is the multiplier on the household's budget constraint. Using (62), EV is

$$\phi = -\sum_{j=1}^{J} \frac{p_j q_j}{y} \Delta \log p_j. \tag{64}$$

That is, equivalent variation is how much prices change weighted by initial budget shares. This is the standard result that many papers have implemented empirically.

7.2. Discrete Choice, Complete Markets

Now, consider the discrete choice model under complete markets as described in Section 3. Here, we show the that with complete markets, we obtain a version of (64).

As we did above, we can compute $V_{p_j}\Big(\mathbf{p}, y\Big)$ and $V_y\Big(\mathbf{p}, y\Big)$:

$$V_{p_j}\Big(\mathbf{p}, y\Big) = -\lambda \int \sum_j \varphi(\boldsymbol{\xi}) x_j(\boldsymbol{\xi}) q_j(\boldsymbol{\xi})\, d\boldsymbol{\xi} = -\lambda \rho_j q_j, \qquad V_y\Big(\mathbf{p}, y\Big) = \lambda \int \varphi(\boldsymbol{\xi}), d\boldsymbol{\xi} = \lambda, \tag{65}$$

where λ is the multiplier on the unified budget constraint across all possible realizations of the taste shock. The final equalities result from arguments similar to those Section 3 — that the state prices are actuarially fair and that we know $q_j(\boldsymbol{\xi})$ is independent of the taste shock $\boldsymbol{\xi}$. Inserting these into (62), we have

$$\phi = -\sum_{j=1}^{J} \frac{\rho_j p_j q_j}{y} \Delta \log p_j. \tag{66}$$

Equivalent variation in the discrete choice model is essentially the same as (64). The insight as well as what makes this work, is complete markets. Complete markets allow households to face a unified budget constraint and thus allow the household to effectively behave and experience welfare gains *as if* it were consuming all goods at once.

The only difference between (64) and (66) is that the choice probability now enters into the share calculation. One interpretation is that from perspective of an individual making a decision, the expected budget share $\frac{\rho_j p_j q_j}{y}$ is the welfare-relevant share. The second interpretation behind (66) is that this is the aggregated budget share across all households in the economy. Some individuals purchase bananas, some purchase apples, but when aggregating budget shares across these discrete choices, they are $\frac{\rho_j p_j q_j}{y}$.

This last observation connects with empirical measurements of equivalent variation and how they are put to practice. Suppose a researcher had scanner or survey data on a *group* of individuals of this type. For example, Del Canto et al. (2023) group together high-school-educated households aged 35-40 and use the CEX to compute the budget shares *of that group* on a collection of goods. One interpretation behind this measurement is that each individual consumes a little of every single good. An alternative interpretation is that individuals are making discrete choices across the different goods and there are complete markets. From this perspective, we have further generalized the interpretation and foundation behind results like the ones in

Deaton (1989). The key restriction, however, is complete markets, and we illustrate this next.

7.3. Discrete Choice, Incomplete Markets

Now, consider the discrete choice model with incomplete markets. Here, we obtain a substantively different formula relative to (66).

Computing $V_{p_j}\Big(\mathbf{p}, y\Big)$ and $V_y\Big(\mathbf{p}, y\Big)$ with incomplete markets gives

$$V_{p_j}\Big(\mathbf{p}, y\Big) = -\int x_j(\boldsymbol{\xi})\lambda_j(\boldsymbol{\xi})q_j(\boldsymbol{\xi})d\boldsymbol{\xi} = -\rho_j \frac{u'(q_j)q_j}{p_j}, \quad V_y\Big(\mathbf{p}, y\Big) = \int \sum_j x_j(\boldsymbol{\xi})\lambda_j(\boldsymbol{\xi})d\boldsymbol{\xi} = \sum_j \rho_j \frac{u'(q_j)}{p_j}.$$

Here, the consequence of incomplete markets is twofold. The cost of a price change is siloed only into states where j is consumed, rather than equalized as in the complete markets case. In other words, the value of a price reduction is not the same across goods. Similarly, the value of income y varies across states, and hence $V_y(\boldsymbol{p}, y)$ is the expected marginal value of income. Combining these terms, we obtain our third expression for equivalent variation:

$$\phi = -\sum_{j=1}^{J} \left(\frac{\rho_j u'(q_j)/p_j}{\sum_k \rho_k u'(q_k)/p_k} \right) \frac{p_j q_j}{y} \Delta \log p_j. \tag{67}$$

With incomplete markets, the formula looks familiar, but with a factor that twists the weighting on budget shares toward states in which the individual has a high marginal value of income.[10] Intuitively, if the individual cannot move resources into states where they like expensive choices (a high $\lambda_j(\boldsymbol{\xi})$), then further increases in prices in these states will be more costly.

An interesting feature of equivalent variation with incomplete markets is that we can connect it with price elasticities of demand. To see this, define ε_j as what one would measure as the price elasticity of demand:

$$\varepsilon_j = -\frac{\partial \log [\rho_j q_j]}{\partial \log p_j} = \underbrace{\left[-\frac{\partial \log \rho_j}{\partial \log p_j} \right]}_{\text{Extensive margin}} + \underbrace{\left[-\frac{\partial \log q_j}{\partial \log p_j} \right]}_{\text{Intensive margin}}. \tag{68}$$

Then assuming that the taste shocks are type 1 extreme value distributed, the price elasticity of demand is

$$\varepsilon_j = \eta\left(1 - \rho_j\right) u'\left(q_j\right) q_j - 1. \tag{69}$$

[10] Inspecting (67) also connects back with our Anderson et al. (1987) example, where the incomplete markets allocation aligned with the complete markets allocation. As one would suspect, the welfare consequences are the same under this preference specification. Specifically, with log preferences $u'(q_j)/p_j = 1/y$ for all j goods and (67) collapses back to the complete markets expression.

Using this in the incomplete markets expression for equivalent variation in (67), we arrive at

$$\phi = -\sum_j \left(\frac{\left(\frac{\rho_j}{1-\rho_j}\right)\left(1+\varepsilon_j\right)}{\sum_k \left(\frac{\rho_k}{1-\rho_k}\right)\left(1+\varepsilon_j\right)} \right) \Delta \log p_j. \tag{70}$$

This expression is important for the following reason: to a first order, elasticities of demand appear in the computation of equivalent variation *under incomplete markets*. The idea behind this relates back to our discussion of the incomplete markets allocation. In that allocation, marginal utility is not equated across goods choices, and thus changes in prices and the substitution that they induce have first-order welfare impacts. In contrast, with complete markets, these margins are equated and equivalent variation aligns with the conventional wisdom that budget shares are the sufficient to characterize the welfare effects of price changes.

This result is worth contrasting with Auer et al.'s (2022) very nice presentation regarding the welfare effects of price changes. Starting from a standard continuous choice problem like in Section 7.1, their first order components are like the ones in (62), and elasticities of demand show up only in *second order* components in equivalent variation. Their contribution is to show that these second order components are large and systematically vary across rich and poor households. Our contribution is to show that with discrete choice with incomplete markets, elasticities of demand show up to the *first order* and that the role that elasticities of demand play (or not) depends on assumptions about market incompleteness.

8. Conclusion

We started from two observations. First, discrete choice models are a powerful and simple framework for thinking about choice problems. Second, in seminars or papers about the welfare impacts from various shocks in this class of models, there was a nagging question: what are properties of the allocations from which welfare is evaluated? Given the additivity of the shock, the answers seemed not interesting. This was not the case. These economies are generically inefficient. However, we then showed how the complete markets and Pareto efficient allocations take simple intuitive forms and how they can be put to use in a broad array of settings.

The purpose of this paper is not to take a stand, per se, on the extent of market incompleteness—on the contrary. Discrete choice models are often used to (i) account for heterogeneity in behavior in a parsimonious and flexible way and (ii) evaluate a specific policy— for example, a place-based policy or merger. But how does one evaluate the welfare gains from the policy versus the welfare effects that arise because of how heterogeneity is introduced? This is what our complete markets formulation achieved. And we hope that by providing an appropriate benchmark to isolate and evaluate the effects of the specific policy, it is useful.

There are several areas for future research. One is the extension to dynamic frameworks. In the background, and as alluded to at various points in the text, we very much thought of the static problem as being a representation of dynamic environments. That's not quite the case, though, because the choice today often influences values in the future, and thus efficient and complete markets allocations may be different than what we characterized. The characterization of socially optimal rural-urban migration flows in Lagakos et al. (2023) is one example; Donald et al. (2023a) is another example, studying optimal policy in dynamic spatial models.

The second question regards the role of partial / self-insurance and discrete choices. The standard incomplete markets model, where households can imperfectly insure against labor income risk, has proven to be a very useful laboratory for thinking about household heterogeneity and consumption dynamics. Given our results, we think that there are many interesting questions for research regarding how partial insurance influences discrete choices such as where one lives, which good you buy or employer you work for.

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Appendix

A. Appendix: Incomplete Markets

Here walk through the arguments regarding how we solve the agents problem in incomplete markets. Let us state the problem:

$$\max_{c_j^i(\boldsymbol{\xi},\theta),\ q_j^i(\boldsymbol{\xi},\theta),\ x_j^i(\boldsymbol{\xi},\theta)} \int_{\boldsymbol{\xi}} \sum_j x_j^i\big(\boldsymbol{\xi},\theta\big)\Big\{u\big[c^i(\boldsymbol{\xi},\theta),q_j^i(\boldsymbol{\xi},\theta);j,\theta\big]+\xi_j^i\Big\}g\big(\boldsymbol{\xi};\theta\big)\,d\boldsymbol{\xi}\,, \tag{71}$$

$$\text{subject to:} \qquad \big[\ \lambda^i(\boldsymbol{\xi},\theta)\ \big]: \quad \sum_j x_j^i(\boldsymbol{\xi},\theta)\big[c^i(\boldsymbol{\xi},\theta)+p_j q_j^i(\boldsymbol{\xi},\theta)\big] \leq\ W(\theta), \quad \text{for all } \boldsymbol{\xi}. \tag{72}$$

A household chooses consumption quantities, and commodity choices for every possible state subject to the constraint is that consumption equals the households endowment — state by state. The Lagrangian associated with this problem is

$$\mathcal{L} = \max_{c_j(\boldsymbol{\xi}),\ q_j(\boldsymbol{\xi}),\ x_j(\boldsymbol{\xi})} \int_{\boldsymbol{\xi}} \sum_j x_j^i(\boldsymbol{\xi},\theta)\Big\{u\big[c^i(\boldsymbol{\xi},\theta),q_j^i(\boldsymbol{\xi},\theta);j,\theta\big]+\xi_j^i\Big\}g(\boldsymbol{\xi};\theta)d\boldsymbol{\xi}, \tag{73}$$

$$+\int_{\boldsymbol{\xi}} \lambda^i(\boldsymbol{\xi},\theta)\Big\{W(\theta)-\sum_j x_j^i(\boldsymbol{\xi},\theta)\big[c^i(\boldsymbol{\xi},\theta)+p_j q_j^i(\boldsymbol{\xi},\theta)\big]\Big\}g(\boldsymbol{\xi},\theta)d\boldsymbol{\xi}. \tag{74}$$

The strategy is to characterize necessary conditions that determine consumption and then characterize the rule which determines which good to chose.

The first order condition for consumption is

$$x_j^i(\boldsymbol{\xi},\theta)u_{q_j}\big[c^i(\boldsymbol{\xi},\theta),q_j^i(\boldsymbol{\xi},\theta);j,\theta\big]g(\boldsymbol{\xi},\theta) = \lambda^i(\boldsymbol{\xi},\theta)x_j^i(\boldsymbol{\xi},\theta)p_j g(\boldsymbol{\xi},\theta), \tag{75}$$

Notice that we do not have to take a stand on the value of $x_j^i(\boldsymbol{\xi},\theta)$. To see this notice that there are two cases: (i) the first order condition is trivially satisfied as $x_j^i(\boldsymbol{\xi},\theta)=0$ or (ii) $x_j^i(\boldsymbol{\xi},\theta)=1$. Furthermore, notice how the pdf for the taste shock shows up on both the left and the right hand side, hence, we can canceling terms giving

$$u_{q_j}\big[c^i(\boldsymbol{\xi},\theta),q_j^i(\boldsymbol{\xi},\theta);j,\theta\big] = \lambda^i(\boldsymbol{\xi},\theta)p_j. \tag{76}$$

Then similarly for the non-differentiated good

$$x_j^i(\boldsymbol{\xi},\theta)u_c\big[c^i(\boldsymbol{\xi},\theta),q_j^i(\boldsymbol{\xi},\theta);j,\theta\big]g(\boldsymbol{\xi},\theta) = \lambda^i(\boldsymbol{\xi},\theta)x_j^i(\boldsymbol{\xi},\theta)g(\boldsymbol{\xi},\theta), \tag{77}$$

and then canceling terms we have

$$u_c\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] = \lambda^i(\boldsymbol{\xi},\theta). \tag{78}$$

Then inspecting these conditions we make several observations. First, all that matters for the multiplier is the choice, not the realization $\boldsymbol{\xi}$, since any $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ that lead to the same choice have the same budget constraint. Second, and because there is no dependence on the left-hand side or right-hand side on the shock, we can drop the dependence of consumption on the realization $\boldsymbol{\xi}$. However, it does depend upon the choice j. Third, we can drop the indexing of the multiplier by i. The argument is that the only thing that might make i households different are the realizations $\boldsymbol{\xi}$, but now the consumption allocations for all i don't depend upon the realization, endowments are the same across i, so the multiplier does not depend upon i. These arguments imply that the first order conditions characterizing consumption allocations are

$$u_c\big[c(j,\theta), q_j(\theta); j,\theta\big] = \lambda_j(\theta) \quad \text{and} \quad u_{q_j}\big[c(j,\theta), q_j(\theta); j,\theta\big] = \lambda_j(\theta)p_j. \tag{79}$$

The next step is to characterize the $x_j(\boldsymbol{\xi},\theta)$ that is the rule mapping the realization of the taste shock into which good should be chosen. We do this incrementally on the Lagrangian by thinking through which j gives the most utility for a given $\boldsymbol{\xi}$). So fix a realization, $\boldsymbol{\xi}$, then compare utility across those events in the Lagrangian...

$$\big[u\big[c^i(\boldsymbol{\xi},\theta), q^i_1(\boldsymbol{\xi},\theta); 1,\theta\big] + \xi^i_1\big]g(\boldsymbol{\xi},\theta) + \lambda^i(\boldsymbol{\xi},\theta)g(\boldsymbol{\xi},\theta)\big[W(\theta) - c^i(\boldsymbol{\xi},\theta) - p_1 q^i_1(\boldsymbol{\xi},\theta)\big] \quad \text{vs.} \tag{80}$$

$$\big[u\big[c^i(\boldsymbol{\xi},\theta), q^i_2(\boldsymbol{\xi},\theta); 2,\theta\big] + \xi^i_2\big]g(\boldsymbol{\xi},\theta) + \lambda^i(\boldsymbol{\xi},\theta)g(\boldsymbol{\xi},\theta)\big[W(\theta) - c^i(\boldsymbol{\xi},\theta) - p_2 q^i_2(\boldsymbol{\xi},\theta)\big] \dots \tag{81}$$

Now the budget constraint always binds for every realization $\boldsymbol{\xi}$ independent of the good chosen. This implies that the second terms in the comparison are always zero. Then inserting our arguments about how consumption, the rule describing which good to consume is

$$x^i_j(\boldsymbol{\xi},\theta) = \begin{cases} 1, & \text{if } u\big[c(j,\theta), q_j(\theta); j,\theta\big] + \xi^i_j \quad \geq \\ & \qquad \max\limits_{j'} \left\{ u\big[c(j',\theta), q_{j'}(\theta); j',\theta\big] + \xi^i_{j'} \right\} \\ & \\ 0. & \text{otherwise} \end{cases} \tag{82}$$

As emphasized in the text, this is the same choice rule that is the starting point from the literature: see what the realized shocks are and then the utility associated with consumption amongst the different choices and then chose the one that delivers highest utility. however, we did not employ an ex-post approach, but an ex-ante approach which, with incomplete markets, delivers the same choice rule.

B. Appendix: Complete Markets

Here walk through several arguments. First, we state the households problem and then provide an argument that the complete markets equilibrium is Pareto efficient. Then the second is a characterization of the necessary conditions behind the households optimization problem.

2.1. The Households Problem

Here we state the households problem. It is

$$\max_{a_j(\boldsymbol{\xi}),\ c(\boldsymbol{\xi}),\ q_j(\boldsymbol{\xi}),\ x_j(\boldsymbol{\xi})} \int_{\boldsymbol{\xi}} \sum_j x^i_j(\boldsymbol{\xi},\theta)\Big\{u\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] + \xi^i_j\Big\}g(\boldsymbol{\xi};\theta)d\boldsymbol{\xi}, \qquad \text{subject to:} \tag{83}$$

$$\sum_j x^i_j(\boldsymbol{\xi},\theta)\big[p_o c^i(\boldsymbol{\xi},\theta) + p_j q^i_j(\boldsymbol{\xi},\theta)\big] \leq\ W(\theta) + a^i(\boldsymbol{\xi},\theta) \quad \forall\boldsymbol{\xi}, \tag{84}$$

$$\int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta)a^i(\boldsymbol{\xi},\theta)d\boldsymbol{\xi} = 0, \tag{85}$$

where $\varphi(\boldsymbol{\xi},\theta)$ are the actuarially fair state prices. A household chooses contingent claims, consumption quantities, and commodity choices for every possible state subject to two constraints. The first constraint is that consumption equals endowments and amount of insurance purchased — state by state. The second constraint says that the household's net asset position must be zero.

Then rewrite the problem in (83) by substituting in the budget constraints from (84) into the constraint for assets (85). The resulting problem is

$$\max_{a_j(\boldsymbol{\xi}),\ c_j(\boldsymbol{\xi}),\ q_j(\boldsymbol{\xi}),\ x_j(\boldsymbol{\xi})} \int_{\boldsymbol{\xi}} \sum_j x^i_j(\boldsymbol{\xi},\theta)\Big\{u\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] + \xi^i_j\Big\}g(\boldsymbol{\xi};\theta)d\boldsymbol{\xi}, \qquad \text{subject to:} \tag{86}$$

$$[\ \lambda^i(\theta)\ ] \qquad \int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta)\Big\{W(\theta) - \sum_j x^i_j(\boldsymbol{\xi},\theta)\big[c^i(\boldsymbol{\xi},\theta) + p_j q^i_j(\boldsymbol{\xi},\theta)\big]\Big\}d\boldsymbol{\xi} = 0. \tag{87}$$

where the new constraint has the interpretation of the "lifetime" budget constraint and now there is just one multiplier $\lambda^i(\theta)$ on this one constraint unlike the incomplete markets problem where there are state by state multipliers. Then from here we can define the Complete Markets

Equilibrium:

Definition 3 (The Complete Markets Equilibrium.) *A complete markets equilibrium are allocations $c^i(\boldsymbol{\xi},\theta)$, $q^i_j(\boldsymbol{\xi},\theta)$, $a^i(\boldsymbol{\xi},\theta)$, $x^i_j(\boldsymbol{\xi},\theta)$ for each i and type θ; goods prices p_j and state prices $\varphi(\boldsymbol{\xi},\theta)$ such that*

i The consumption allocations ($c^i(\boldsymbol{\xi},\theta)$, $q^i_j(\boldsymbol{\xi},\theta)$, and $x^i_j(\boldsymbol{\xi},\theta)$) and asset positions $a^i(\boldsymbol{\xi},\theta)$ satisfy the household's problem in (86);

ii State prices satisfy (12);

iii Goods and asset markets clear.

2.2. Proof of First Welfare Theorem

Given our definition of a Complete Markets equilibrium, we argue that it is Pareto efficient. To establish our argument, first note that any alternative allocation $\{ \widetilde{c}^i(\boldsymbol{\xi},\theta)$, $\widetilde{q}^i_j(\boldsymbol{\xi},\theta)$, $\widetilde{a}^i(\boldsymbol{\xi},\theta)$, $\widetilde{x}_ji(\boldsymbol{\xi},\theta)\}$ that is preferred by agent i (in the sense that it provides higher utility than (16)) must imply that

$$\int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta)\Big\{W(\theta) - \sum_j \widetilde{x}^i_j(\boldsymbol{\xi},\theta)\big[\widetilde{c}^i(\boldsymbol{\xi},\theta) + p_j\widetilde{q}^i_j(\boldsymbol{\xi},\theta)\big]\Big\}d\boldsymbol{\xi} < 0. \tag{88}$$

In words, this alternative and preferred allocation is not budget feasible. Now consider an alternative allocation $\widetilde{c}^i(\boldsymbol{\xi},\theta)$, $\widetilde{q}^i_j(\boldsymbol{\xi},\theta)$, $\widetilde{a}^i(\boldsymbol{\xi},\theta)$, $\widetilde{x}_ji(\boldsymbol{\xi},\theta)$ for all i and θ types that is (i) feasible and (ii) is preferred by all i and θ types. Feasibility implies that

$$\sum_\theta \int_0^{\mu(\theta)} y^i_o(\theta)di = \sum_\theta \int_{\boldsymbol{\xi}} \int_0^{\mu(\theta)} c^i(\boldsymbol{\xi},\theta)g(\boldsymbol{\xi},\theta)did\boldsymbol{\xi} \tag{89}$$

$$\sum_\theta \int_0^{\mu(\theta)} y^i_j(\theta)di = \sum_\theta \int_{\boldsymbol{\xi}} \int_0^{\mu(\theta)} \widetilde{x}_ji(\boldsymbol{\xi},\theta)q^i_j(\boldsymbol{\xi},\theta)g(\boldsymbol{\xi};\theta)did\boldsymbol{\xi}, \forall j \tag{90}$$

then combine the outside good and differentiated good feasibility constraints evaluated at prices

$$\sum_{\theta}\int_{0}^{\mu(\theta)} y_o^i(\theta)di + \sum_{\theta}\int_{0}^{\mu(\theta)} p_j y_j^i(\theta)di = \tag{91}$$

$$\sum_{\theta}\int_{\boldsymbol{\xi}}\int_{0}^{\mu(\theta)} c^i(\boldsymbol{\xi},\theta)g(\boldsymbol{\xi},\theta)did\boldsymbol{\xi} + \sum_{j}\sum_{\theta}\int_{\boldsymbol{\xi}}\int_{0}^{\mu(\theta)} \widetilde{x}_j i(\boldsymbol{\xi},\theta)p_j q_j^i(\boldsymbol{\xi},\theta)g(\boldsymbol{\xi};\theta)did\boldsymbol{\xi} \tag{92}$$

and then rearrange everything so one can see it i by i so we have

$$\sum_{\theta}\int_{0}^{\mu(\theta)} W(\theta)di = \sum_{\theta}\int_{\boldsymbol{\xi}}\int_{0}^{\mu(\theta)} \left[\sum_{j} x_j^{'i}(\boldsymbol{\xi},\theta)\left[\widetilde{c}^i(\boldsymbol{\xi},\theta) + p_j q_j^{'i}(\boldsymbol{\xi},\theta)\right]g(\boldsymbol{\xi};\theta)did\boldsymbol{\xi} \tag{93}$$

then make the observation that asset trades occur at actuarially fair prices and then change the order of integration giving

$$\sum_{\theta}\int_{0}^{\mu(\theta)} W(\theta)di = \sum_{\theta}\int_{0}^{\mu(\theta)}\int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta)\left[\sum_{j} x_j^{'i}(\boldsymbol{\xi},\theta)\left[\widetilde{c}^i(\boldsymbol{\xi},\theta) + p_j q_j^{'i}(\boldsymbol{\xi},\theta)\right]d\boldsymbol{\xi}di. \tag{94}$$

so feasibility then implies that each households budget constraint is satisfied. But if the allocation is preferred than

$$\sum_{\theta}\int_{0}^{\mu(\theta)} W(\theta)di = \sum_{\theta}\int_{0}^{\mu(\theta)}\int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta)\left[\sum_{j} x_j^{'i}(\boldsymbol{\xi},\theta)\left[\widetilde{c}^i(\boldsymbol{\xi},\theta) + p_j q_j^{'i}(\boldsymbol{\xi},\theta)\right]d\boldsymbol{\xi}di > \sum_{\theta}\int_{0}^{\mu(\theta)} W(\theta)di \tag{95}$$

where the right-hand side inequality follows from (88) and we have our contradiction. Thus, an allocation that satisfies the definition of a complete markets equilibrium is a Pareto efficient allocation.

2.3. Solving the Households Problem

Here walk through the arguments regarding how we characterize the agents problem in complete markets. The households Lagrangian associated with the complete markets problem is

$$\mathcal{L} = \max_{c_j(\boldsymbol{\xi}),\, q_j(\boldsymbol{\xi}),\, x_j(\boldsymbol{\xi})} \int_{\boldsymbol{\xi}} \sum_j x^i_j(\boldsymbol{\xi},\theta)\Big\{ u\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j, \theta\big] + \xi^i_j \Big\} g(\boldsymbol{\xi};\theta) d\boldsymbol{\xi}, \tag{96}$$

$$+ \lambda^i(\theta) \int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta) \Big\{ W(\theta) - \sum_j x^i_j(\boldsymbol{\xi},\theta)\big[c^i(\boldsymbol{\xi},\theta) + p_j q^i_j(\boldsymbol{\xi},\theta)\big] \Big\} d\boldsymbol{\xi}. \tag{97}$$

The first order condition for consumption is

$$x^i_j(\boldsymbol{\xi},\theta) u_{q_j}\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] g(\boldsymbol{\xi},\theta) = \lambda^i(\theta)\varphi(\boldsymbol{\xi},\theta) x^i_j(\boldsymbol{\xi},\theta) p_j. \tag{98}$$

Now insert the fact that the state prices are actuarially fair and then canceling terms we have

$$u_{q_j}\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] = \lambda^i(\theta) p_j. \tag{99}$$

Then similarly for the non-differentiated good

$$x^i_j(\boldsymbol{\xi},\theta) u_c\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] g(\boldsymbol{\xi},\theta) = \lambda^i(\theta)\varphi(\boldsymbol{\xi},\theta) x^i_j(\boldsymbol{\xi},\theta), \tag{100}$$

and then with actuarially fair state prices and canceling terms we have

$$u_c\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] = \lambda^i(\theta). \tag{101}$$

Then the same arguments applied in the incomplete markets problem regarding how certain indexing can be dropped apply here as well. These arguments imply that the first order conditions characterizing consumption allocations are

$$u_c\big[c(j,\theta), q_j(\theta); j,\theta\big] = \lambda(\theta) \quad \text{and} \quad u_{q_j}\big[c(j,\theta), q_j(\theta); j,\theta\big] = \lambda(\theta) p_j. \tag{102}$$

Then this condition leads to a risk-sharing type result where marginal utilities are equated across *all* events — both explicitly as there is no dependence upon $\boldsymbol{\xi}$ and then implicity as there is no dependence upon the choice.

We can follow the same arguments in the incomplete markets problem to characterize $x_j(\boldsymbol{\xi},\theta)$.

So fix an event, then compare utility across those events in the Lagrangian...

$$\big[u\big[c^i(\boldsymbol{\xi},\theta),q_1^i(\boldsymbol{\xi},\theta);1,\theta\big]+\xi_1^i\big]g(\boldsymbol{\xi},\theta)+\lambda^i(\theta)\varphi(\boldsymbol{\xi},\theta)\big[W(\theta)-c^i(\boldsymbol{\xi},\theta)-p_1q_1^i(\boldsymbol{\xi},\theta)\big]\quad \text{vs.} \tag{103}$$

$$\big[u\big[c^i(\boldsymbol{\xi},\theta),q_2^i(\boldsymbol{\xi},\theta);2,\theta\big]+\xi_2^i\big]g(\boldsymbol{\xi},\theta)+\lambda^i(\theta)\varphi(\boldsymbol{\xi},\theta)\big[W(\theta)-c^i(\boldsymbol{\xi},\theta)-p_2q_2^i(\boldsymbol{\xi},\theta)\big],\ldots \tag{104}$$

Then using the observation that state prices are actuarially fair means the comparison reduces to:

$$\big[u\big[c^i(\boldsymbol{\xi},\theta),q_1^i(\boldsymbol{\xi},\theta);1,\theta\big]+\xi_1^i\big]-\lambda^i(\theta)\big[c^i(\boldsymbol{\xi},\theta)+p_1q_1^i(\boldsymbol{\xi},\theta)\big]\quad \text{vs.} \tag{105}$$

$$\big[u\big[c^i(\boldsymbol{\xi},\theta),q_2^i(\boldsymbol{\xi},\theta);2,\theta\big]+\xi_2^i\big]-\lambda^i(\theta)\big[c^i(\boldsymbol{\xi},\theta)+p_1q_2^i(\boldsymbol{\xi},\theta)\big]\ldots. \tag{106}$$

At this point, there are still lots of difficulties, specifically how are the quantities varying with the realization of the shock. This problem is solved by inserting the observations about how consumption does not depend upon $\boldsymbol{\xi}$ which gives the following choice rule

$$x_j^i(\boldsymbol{\xi},\theta)=\begin{cases}1, & \text{if } u\big[c(j,\theta),q_j(\theta);j,\theta\big]+\xi_j^i-\lambda(\theta)\big[c(j,\theta)+p_jq_j(\theta)\big] \;\geq \\ & \quad \max\limits_{j'}\Big\{\,u\big[c(j',\theta),q_{j'}(\theta);j',\theta\big]+\xi_{j'}^i-\lambda(\theta)\big[c(j',\theta)+p_{j'}q_{j'}(\theta)\big]\,\Big\} \\ & \\ 0. & \text{otherwise}\end{cases} \tag{107}$$

The key novelty are the $\lambda(\theta)$ term which is not in the rule for the standard allocation (equation 9). This additional term reflects the cost of choosing that commodity on the consolidated budget constraint.

The final property we consider are the asset positions. Interestingly, a households positions are only contingent the choice, not the particular shock realization $\boldsymbol{\xi}$. This follows from and the properties of consumption discussed above and the budget constraint, so

$$a(j,\theta)=W(\theta)-\big[c(j,\theta)+p_jq_j(\theta)\big]. \tag{108}$$

Finally, notice that one can insert (108) into the choice rule giving a very simple representation

$$x_j^i(\boldsymbol{\xi}, \theta) = \begin{cases} 1, & \text{if } u\big[c(j,\theta), q_j(\theta); j, \theta\big] + \xi_j^i - \lambda(\theta)a(j,\theta) \;\; \geq \\ & \qquad \max_{j'} \Big\{ u\big[c(j',\theta), q_{j'}(\theta); j', \theta\big] + \xi_{j'}^i - \lambda(\theta)a(j',\theta) \Big\} \\ & \\ 0, & \text{otherwise} \end{cases} \tag{109}$$

which gives interpretation on the last terms as the cost of purchasing insurance associated with the choices j. If we impose the type 1 extreme value distributional assumption and integrate up we obtain the efficient choice probabilities:

$$\exp\left(\frac{u\big[c(j,\theta), q_j(\theta), j, \theta\big] - \lambda(\theta)a(j,\theta)}{\eta_\theta}\right) \Bigg/ \sum_{j'} \exp\left(\frac{u\big[c(j',\theta), q_{j'}(\theta), j', \theta\big] - \lambda(\theta)a(j',\theta)}{\eta_\theta}\right). \tag{110}$$

Below we summarize these results.

Proposition 7 (Complete Markets Allocations) *The following conditions characterize the complete markets allocations:*

1. *Consumption allocations must satisfy*

$$u_c\big[c(\theta,j), q_j(\theta); j, \theta\big] = \lambda(\theta) \quad \textit{and} \quad u_{q_j}\big[c(\theta,j), q_j(\theta); j, \theta\big] = \lambda(\theta)p_j,$$

2. *The commodity choice rule is*

$$x_j^i(\boldsymbol{\xi}, \theta) = \begin{cases} 1, & \textit{if } u\big[c(j,\theta), q_j(\theta); j, \theta\big] + \xi_j^i - \lambda(\theta)a(j,\theta) \;\; \geq \\ & \qquad \max_{j'} \Big\{ u\big[c(j',\theta), q_{j'}(\theta); j', \theta\big] + \xi_{j'}^i - \lambda(\theta)a(j',\theta) \Big\} \\ & \\ 0, & \textit{otherwise} \end{cases}$$

3. *"Arrow Vouchers." Asset positions are given by*

$$a(j,\theta) = W(\theta) - \big[c(j,\theta) + p_j q_j(\theta)\big],$$

which are contingent only on the choice j, not the taste shock $\boldsymbol{\xi}$. The state prices for the

Arrow Vouchers are the choice probabilities which with the Type 1 EV assumption are

$$\exp\left(\frac{u\big[c(j,\theta),q_j(\theta),j,\theta\big]-\lambda(\theta)a(j,\theta)}{\eta_\theta}\right)\Bigg/\sum_{j'}\exp\left(\frac{u\big[c(j',\theta),q_{j'}(\theta),j',\theta\big]-\lambda(\theta)a(j',\theta)}{\eta_\theta}\right).$$

C. Appendix: The Pareto Problem

The Pareto problem is to find allocations that maximize somebody's utility subject to a resource constraint and that the allocation must respect given utility levels for all other households in the economy. The resulting allocation is then by definition of the problem a Pareto efficient allocation.

Utility for household i of type θ is

$$V^i(\theta)=\int_{\boldsymbol{\xi}}\sum_j x^i_j(\boldsymbol{\xi},\theta)\Big\{u\big[c^i(\boldsymbol{\xi},\theta),q^i_j(\boldsymbol{\xi},\theta);j,\theta\big]+\xi^i_j\Big\}g(\boldsymbol{\xi};\theta)d\boldsymbol{\xi}, \tag{111}$$

and then we will index all other households by the label k, θ' with given utility level $V^k(\theta')$. Then the Pareto problem is

$$\max_{c_j(\boldsymbol{\xi},\theta),q_j(\boldsymbol{\xi},\theta),x_j(\boldsymbol{\xi},\theta)\ \forall i,k,\theta'}\int_{\boldsymbol{\xi}}\sum_j x^i_j(\boldsymbol{\xi},\theta)\Big\{u\big[c^i(\boldsymbol{\xi},\theta),q^i_j(\boldsymbol{\xi},\theta);j,\theta\big]+\xi^i_j\Big\}g(\boldsymbol{\xi};\theta)d\boldsymbol{\xi} \tag{112}$$

subject to the resource constraints:

$$[\,\Lambda_o\,]\quad \sum_{\theta'}\int_0^{\mu(\theta')}y^k_o(\theta')dk\geq\sum_{\theta'}\int_\xi\int_k x^k_j(\boldsymbol{\xi},\theta')c^k_j(\boldsymbol{\xi},\theta')\,dk\,g(\boldsymbol{\xi},\theta')d\boldsymbol{\xi}, \tag{113}$$

$$[\,\Lambda_j\,]\quad \sum_{\theta'}\int_0^{\mu(\theta')}y^k_j(\theta')dk\geq\sum_{\theta'}\int_\xi\int_k x^k_j(\boldsymbol{\xi},\theta')q^k_j(\boldsymbol{\xi},\theta')\,dk\,g(\boldsymbol{\xi},\theta')d\boldsymbol{\xi}\quad\forall j. \tag{114}$$

which says that goods supply must be greater than or equal to goods demand. Associated with these constraints are the Lagrange multiplier Λ_o and Λ_j for each good j. The next constraint is the Pareto constraint:

$$[\,\Upsilon^k(\theta')\,]\quad V^k(\theta')\leq\int_{\boldsymbol{\xi}}\sum_j x^k_j(\boldsymbol{\xi},\theta')\Big\{u\big[c^k(\boldsymbol{\xi},\theta'),q^k_j(\boldsymbol{\xi},\theta');j,\theta'\big]+\xi^k_j\Big\}g(\boldsymbol{\xi},\theta')d\boldsymbol{\xi}\quad\forall k,\theta'\neq i,\theta. \tag{115}$$

This says that at any allocation, it has to deliver utility level $V^k(\theta')$ (or be better) for every k,θ' household. Associated with each of these constraints is the Lagrange multiplier $\Upsilon^k(\theta')$. Putting the problem all together we have

$$\mathcal{L} = \max_{c_j(\boldsymbol{\xi},\theta), q_j(\boldsymbol{\xi},\theta), x_j(\boldsymbol{\xi},\theta)\ \forall i,k,\theta'} \int_{\boldsymbol{\xi}} \sum_j x_j^i(\boldsymbol{\xi},\theta) \Big\{ u\big[c^i(\boldsymbol{\xi},\theta), q_j^i(\boldsymbol{\xi},\theta); j, \theta\big] + \xi_j^i \Big\} g(\boldsymbol{\xi};\theta) d\boldsymbol{\xi} \tag{116}$$

$$+ \Lambda_o \Big[\sum_{\theta'} Y_o(\theta') - \sum_{\theta'} \int_{\xi} \int_k x_j^k(\boldsymbol{\xi},\theta') c_j^k(\boldsymbol{\xi},\theta')\ dk\ g(\boldsymbol{\xi},\theta') d\boldsymbol{\xi} \Big] \tag{117}$$

$$+ \sum_j \Lambda_j \Big[\sum_{\theta'} Y_j(\theta') - \sum_{\theta'} \int_{\xi} \int_k x_j^k(\boldsymbol{\xi},\theta') q_j^k(\boldsymbol{\xi},\theta')\ dk\ g(\boldsymbol{\xi},\theta') d\boldsymbol{\xi} \Big] \tag{118}$$

$$+ \sum_{\theta'} \int_k \Upsilon^k(\theta') \Big[\int_{\boldsymbol{\xi}} \sum_j x_j^k(\boldsymbol{\xi},\theta') \Big\{ u\big[c^k(\boldsymbol{\xi},\theta'), q_j^k(\boldsymbol{\xi},\theta'); j, \theta'\big] + \xi_j^k \Big\} g(\boldsymbol{\xi},\theta') d\boldsymbol{\xi} - V^k(\theta') \Big] dk\ g(\boldsymbol{\xi}) d\boldsymbol{\xi}. \tag{119}$$

As a recap: This planner chooses allocations for everybody: i, θ and all k, θ' to maximize welfare for i, θ given the resource constraint and then the idea that all k, θ' households must be delivered at least $V^k(\theta')$ level of utility.

The next steps derive first order conditions and characterize the rule describing which goods are chosen and under which circumstances. We do this below in steps.

Conditional on a choice, the first order condition for is consumption of variety q_j is

$$x_j^i(\boldsymbol{\xi},\theta) u_{q_j}\big[c^i(\boldsymbol{\xi},\theta), q_j^i(\boldsymbol{\xi},\theta); j, \theta\big] g(\boldsymbol{\xi},\theta) = \Lambda_j x_j^i(\boldsymbol{\xi},\theta) g(\boldsymbol{\xi},\theta) \tag{120}$$

and then canceling terms we have

$$u_{q_j}\big[c^i(\boldsymbol{\xi},\theta), q_j^i(\boldsymbol{\xi},\theta); j, \theta\big] = \Lambda_j. \tag{121}$$

And then a similar condition holds for the non-differentiated commodity

$$x_j^i(\boldsymbol{\xi},\theta) u_c\big[c^i(\boldsymbol{\xi},\theta), q_j^i(\boldsymbol{\xi},\theta); j, \theta\big] g(\boldsymbol{\xi},\theta) = \Lambda_o x_j^i(\boldsymbol{\xi},\theta) g(\boldsymbol{\xi},\theta), \tag{122}$$

and then canceling terms we have

$$u_c\big[c^i(\boldsymbol{\xi},\theta), q_j^i(\boldsymbol{\xi},\theta); j, \theta\big] = \Lambda_o. \tag{123}$$

Then we make the observation that the quantities don't depend upon the particular shock realization $\boldsymbol{\xi}$ since the multipliers don't depend on the shock giving

$$u_c\big[c^i(j,\theta), q^i_j(\theta); j, \theta\big] = \Lambda_o. \tag{124}$$

The next question is then what about k, θ's consumption. For variety j we have

$$\Upsilon^k(\theta')x^k_j(\boldsymbol{\xi},\theta')u_{q_j}\left[c^k(\boldsymbol{\xi},\theta'), q^k_j(\boldsymbol{\xi},\theta'); j, \theta\right]g(\boldsymbol{\xi},\theta') = \Lambda_j x^k_j(\boldsymbol{\xi},\theta')g(\boldsymbol{\xi},\theta'), \tag{125}$$

and then canceling terms gives

$$u_{q_j}\left[c^k(\boldsymbol{\xi},\theta), q^k_j(\boldsymbol{\xi},\theta); j, \theta'\right] = \frac{\Lambda_j}{\Upsilon^k(\theta')}. \tag{126}$$

then re-indexing with the dropping of the choice we have

$$u_{q_j}\left[c^k(j,\theta), q^k_j(\theta); j, \theta'\right] = \frac{\Lambda_j}{\Upsilon^k(\theta')}. \tag{127}$$

And then we have the similar condition for the non-differentiated commodity

$$u_c\big[c^k(j,\theta'), q^k_j(\theta'); j, \theta'\big] = \frac{\Lambda_o}{\Upsilon^k(\theta')}. \tag{128}$$

Similar to above, quantities for the k, θ' guys are set so that marginal utility equals the multiplier on the resource constraint adjusted by the multiplier on the Pareto constraint. The adjustment for the multiplier on the Pareto constraint then has the interpretation as the weight that the planner places on agent k, θ'.

From here we can take ratios of these conditions and arrive at results that any Pareto efficient allocation must satisfy. Specifically

$$\frac{u_{q_j}\left[c^k(j,\theta), q^k_j(\theta); j, \theta'\right]}{u_{q'_j}\left[c^k(j,\theta), q'^k_j(\theta); j', \theta'\right]} = \frac{\Lambda_j}{\Lambda'_j}. \tag{129}$$

and this condition holds for any k, θ' agent (including i, θ), any shock realization, and any ratio of commodity j, j' or o. This says that the ratio of marginal rate of substitution between these commodities, for any person, must equal the shadow cost of those commodities.

The next step is to characterize the commodity choice rule or the $x^i_j(\boldsymbol{\xi},\theta)$s. Like in the cases above, the variational approach can be applied as well. So we just inspect the Lagrangian, choice by choice for the i, θ guy under a particular realization of the shock

$$\big[u\big[c^i(\boldsymbol{\xi},\theta), q_1^i(\boldsymbol{\xi},\theta); 1, \theta\big] + \xi_1^i\big] g(\boldsymbol{\xi},\theta) - \Bigg[\Lambda_o c^i(\boldsymbol{\xi},\theta) + \Lambda_1 q_1^i(\boldsymbol{\xi},\theta)\Bigg] g(\boldsymbol{\xi},\theta) \quad \text{vs.} \tag{130}$$

$$\big[u\big[c^i(\boldsymbol{\xi},\theta), q_2^i(\boldsymbol{\xi},\theta); 2, \theta\big] + \xi_2^i\big] g(\boldsymbol{\xi},\theta) - \Bigg[\Lambda_o c^i(\boldsymbol{\xi},\theta) + \Lambda_2 q_2^i(\boldsymbol{\xi},\theta)\Bigg] g(\boldsymbol{\xi},\theta), \ldots \tag{131}$$

Then canceling the densities and inserting the result that the consumption does not depend upon $\boldsymbol{\xi}$

$$x_j^i(\boldsymbol{\xi},\theta) = \begin{cases} 1, & \text{if } \ u\big[c^i(j,\theta), q_j^i(\theta); j, \theta\big] + \xi_j^i - \Bigg[\Lambda_o c^i(j,\theta) + \Lambda_j q_j^i(\theta)\Bigg] \ \geq \\ & \qquad \max_{j'} \Bigg\{ \ u\big[c^i(j',\theta), q_{j'}^i(\theta); j', \theta\big] + \xi_{j'}^i - \Bigg[\Lambda_o c^i(j',\theta) + \Lambda_{j'} q_{j'}^i(\theta)\Bigg]\Bigg\} \\ \\ 0, & \text{otherwise} \end{cases} \tag{132}$$

Now the interesting case is the k, θ' guy. Again, use the same variational argument with

$$\Upsilon^k(\theta')\big[u\big[c^k(\boldsymbol{\xi},\theta'), q_1^k(\boldsymbol{\xi},\theta'); 1, \theta'\big] + \xi_1^k\big] g(\boldsymbol{\xi},\theta') - \Bigg[\Lambda_o c^k(\boldsymbol{\xi},\theta') + \Lambda_1 q_1^k(\boldsymbol{\xi},\theta')\Bigg] g(\boldsymbol{\xi},\theta') \quad \text{vs.} \tag{133}$$

$$\Upsilon^k(\theta')\big[u\big[c^k(\boldsymbol{\xi},\theta'), q_2^k(\boldsymbol{\xi},\theta'); 2, \theta'\big] + \xi_2^k\big] g(\boldsymbol{\xi},\theta') - \Bigg[\Lambda_o c^k(\boldsymbol{\xi},\theta') + \Lambda_2 q_2^k(\boldsymbol{\xi},\theta')\Bigg] g(\boldsymbol{\xi},\theta') \ldots \tag{134}$$

Then we follow the same steps as with the i case. Terms cancel, insert the result that the consumption allocation does not depend upon the shock, then divide through by the multiplier $\Upsilon^k(\theta')$. This then gives rise to the following commodity choice rule:

$$x_j^k(\boldsymbol{\xi},\theta') = \begin{cases} 1, & \text{if } \ u\big[c^k(j,\theta'), q_j^k(\theta'); j, \theta'\big] + \xi_j^k - \frac{1}{\Upsilon^k(\theta')}\Bigg[\Lambda_o c^k(j,\theta') + \Lambda_j q_j^k(\theta')\Bigg] \ \geq \\ & \qquad \max_{j'} \Bigg\{ \ u\big[c^k(j',\theta'), q_{j'}^k(\theta'); j', \theta'\big] + \xi_{j'}^k - \frac{1}{\Upsilon^k(\theta')}\Bigg[\Lambda_o c^k(j',\theta') + \Lambda_{j'} q_{j'}^k(\theta')\Bigg]\Bigg\} \\ \\ 0, & \text{otherwise} \end{cases} \tag{135}$$

which holds for all k, θ' households. And note that this rule is equivalent to the i, θ household by setting the Υ term to one. An interesting step is to notice how the ratio of the multipliers then connects with the marginal utility of consumption giving the same exact choice rule as with the i guy

$$x_j^k(\boldsymbol{\xi},\theta') = \begin{cases} 1, & \text{if } \; u\big[c^k(j,\theta'), q_j^k(\theta'); j, \theta'\big] + \xi_j^k - \Big[u_c^k(j,\theta')c^k(j,\theta') + u_{q_j}^k(\theta')q_j^k(\theta')\Big] \;\; \geq \\ & \qquad \max_{j'} \Big\{ \; u\big[c^k(j',\theta'), q_{j'}^k(\theta'); j', \theta'\big] + \xi_{j'}^k - \Big[u_c^k(j,\theta')c^k(j',\theta') + u_{q_{j'}}^k(\theta')q_{j'}^k(\theta')\Big]\Big\} \\ \\ 0, & \text{otherwise} \end{cases} \tag{136}$$

And this rule then looks exactly the same as for the i, θ agent. One interesting feature of this characterization is that the welfare weights (or the $\Upsilon^k(\theta')$) don't directly show up. What is going on is that welfare weights determine how much or little consumption k, θ' person receives, and thus their marginal utility for the commodity. Then these marginal utilities appropriately determines the social cost of choosing a particular variety.

Proposition 8 summarizes the result below.

Proposition 8 (Pareto Efficient Allocations) *Given utility levels $V^k(\theta')$ for all $k, \theta' \neq i, \theta$, a **Pareto efficient allocation** is consumption allocations and commodity choice rules $c_j^i(\boldsymbol{\xi}, \theta)$, $q_j^i(\boldsymbol{\xi}, \theta)$, $x_j^i(\boldsymbol{\xi}, \theta)$ and i, θ and for all other all k, θ', that solve the problem (30) subject to resource constraints (31, 32) and the Pareto constraint in (33).*

The following conditions characterize Pareto efficient allocations:

1. *For agent i, θ, consumption allocations must satisfy*

$$u_c\big[c^i(j,\theta), q_j^i(\theta); j, \theta\big] = \Lambda_o \quad \textit{and} \quad u_{q_j}\big[c^i(j,\theta), q_j^i(\theta); j, \theta\big] = \Lambda_j.$$

2. *For agent k', θ', consumption allocations must satisfy:*

$$\Upsilon^k(\theta')\; u_c\big[c^k(j,\theta), q_j^k(\theta); j, \theta\big] = \Lambda_o \quad \textit{and} \quad \Upsilon^k(\theta')\; u_{q_j}\big[c^k(j,\theta), q_j^k(\theta); j, \theta\big] = \Lambda_j.$$

3. *The commodity choice rule is*

$$x_j^k(\boldsymbol{\xi},\theta') = \begin{cases} 1, & \textit{if } \; u\big[c^k(j,\theta'), q_j^k(\theta'); j, \theta'\big] + \xi_j^k - \frac{1}{\Upsilon^k(\theta')}\Big[\Lambda_o c^k(j,\theta') + \Lambda_j q_j^k(\theta')\Big] \;\; \geq \\ & \qquad \max_{j'} \Big\{ \; u\big[c^k(j',\theta'), q_{j'}^k(\theta'); j', \theta'\big] + \xi_{j'}^k - \frac{1}{\Upsilon^k(\theta')}\Big[\Lambda_o c^k(j',\theta') + \Lambda_{j'} q_{j'}^k(\theta')\Big]\Big\} \\ \\ 0, & \textit{otherwise} \end{cases}$$

which holds for all k, θ' households, and for the i, θ household set the Υ term to one.

We now have a basis for the *Second Welfare Theorem.* This is immediate from comparing the contents of Proposition 2 and from Complete Markets to Proposition 3. Consumption allocations and choice rules are identical if $\Lambda_0/\Upsilon^k(\theta) = \lambda(\theta)$, and $\Lambda_j/\Upsilon^k(\theta) = \lambda(\theta)p_j$. To align the complete markets allocation to the Pareto efficient allocation, then one simply needs to find the appropriate ex-ante reallocation of resources to attain the correct multipliers on complete markets budget constraints, $\lambda(\theta)$.

D. Appendix: The Social Planning Problem

In this section we solve the problem of a planner who maximizes a linear social welfare function with welfare weights $\psi^k(\theta')$ for all k and θ. Inspection of the conditions of this problem then provides an equivalence between Pareto efficient allocations and maximizing linear social welfare functions under some welfare weights.

The planning problem is the following

$$\max_{c^k(\boldsymbol{\xi},\theta'),q^k_j(\boldsymbol{\xi},\theta'),x^k_j(\boldsymbol{\xi},\theta')} \sum_{\theta'} \int_k \psi^k(\theta') \int_{\boldsymbol{\xi}} \sum_j x^k_j(\boldsymbol{\xi},\theta') \Big\{ u\big[c^k(\boldsymbol{\xi},\theta'), q^k_j(\boldsymbol{\xi},\theta'); j, \theta'\big] + \xi^k_j \Big\} g(\boldsymbol{\xi},\theta') d\boldsymbol{\xi} \, dk, \tag{137}$$

$$\text{subject to:} \quad \sum_{\theta'} Y_j(\theta') \geq \sum_{\theta'} \int_{\xi} \int_k x^k_j(\boldsymbol{\xi},\theta') q^k_j(\boldsymbol{\xi},\theta') \, dk \, g(\boldsymbol{\xi},\theta') d\boldsymbol{\xi} \quad \forall j \tag{138}$$

$$\sum_{\theta'} Y_o(\theta') \geq \sum_{\theta'} \int_{\xi} \int_k \sum_j x^k_j(\boldsymbol{\xi},\theta') c^k(\boldsymbol{\xi},\theta') \, dk \, g(\boldsymbol{\xi},\theta') d\boldsymbol{\xi} \tag{139}$$

Where the social planner chooses consumption and commodities for every event $\boldsymbol{\xi}$ and every k, θ' agent in the economy. These allocations are chosen to maximize social welfare in (137) which is a weighted average of each individual agents expected utility, with the weights being the given social welfare weights $\psi^k(\theta')$. This objective function is maximized subject to the resource constraints.

The Lagrangian associated with this problem is

$$\mathcal{L} = \max_{c^k(\boldsymbol{\xi},\theta'),q_j^k(\boldsymbol{\xi},\theta'),x_j^k(\boldsymbol{\xi},\theta')} \sum_{\theta'} \int_k \psi^k(\theta') \int_{\boldsymbol{\xi}} \sum_j x_j^k(\boldsymbol{\xi},\theta') \Big\{ u\big[c^k(\boldsymbol{\xi},\theta'), q_j^k(\boldsymbol{\xi},\theta'); j, \theta'\big] + \xi_j^k \Big\} dk\ g(\boldsymbol{\xi},\theta') d\boldsymbol{\xi}, \tag{140}$$

$$+ \ \sum_j \Lambda_j \Big[\sum_{\theta'} Y_j(\theta') - \sum_{\theta'} \int_\xi \int_k x_j^k(\boldsymbol{\xi},\theta') q_j^k(\boldsymbol{\xi},\theta')\ dk\ g(\boldsymbol{\xi},\theta') d\boldsymbol{\xi} \Big] \tag{141}$$

$$+ \ \Lambda_o \Big[\sum_{\theta'} Y_o(\theta') - \sum_{\theta'} \int_\xi \int_k \sum_j x_j^k(\boldsymbol{\xi},\theta') c^k(\boldsymbol{\xi},\theta')\ dk\ g(\boldsymbol{\xi},\theta') d\boldsymbol{\xi} \Big]. \tag{142}$$

The first order condition for consumption of commodity j becomes (after canceling terms using the same arguments above)

$$\psi^k(\theta') u_{q_j} \big[c^k(j,\theta'), q_j^k(\theta'); j, \theta\big] = \Lambda_j, \tag{143}$$

and then for the non-differentiated commodity we have

$$\psi^k(\theta') u_c \big[c^k(j,\theta'), q_j^k(\theta'); j, \theta\big] = \Lambda_o. \tag{144}$$

The planner sets social-welfare-weighted marginal utility equal to the shadow cost of consuming that commodity. Inspecting these conditions (143, 144) and (121, 123) shows that for a given Pareto efficient allocation, the multipliers on the Pareto constraint maps directly into the social welfare weights associated with Planning problem.

Now social welfare maximizing choice rule $x_j^k(\boldsymbol{\xi},\theta')$ is

$$\psi^k(\theta')\big[u\big[c^k(\boldsymbol{\xi},\theta'), q_1^k(\boldsymbol{\xi},\theta'); 1, \theta'\big] + \xi_1^k\big] g(\boldsymbol{\xi},\theta') - \Big[\Lambda_o c^k(\boldsymbol{\xi},\theta') + \Lambda_1 q_1^k(\boldsymbol{\xi},\theta')\Big] g(\boldsymbol{\xi},\theta') \quad \text{vs.} \tag{145}$$

$$\psi^k(\theta')\big[u\big[c^k(\boldsymbol{\xi},\theta'), q_2^k(\boldsymbol{\xi},\theta'); 2, \theta'\big] + \xi_2^k\big] g(\boldsymbol{\xi},\theta') - \Big[\Lambda_o c^k(\boldsymbol{\xi},\theta') + \Lambda_2 q_2^k(\boldsymbol{\xi},\theta')\Big] g(\boldsymbol{\xi},\theta') \ldots \tag{146}$$

Then terms cancel, insert the result that the consumption allocation does not depend upon the shock, then divide through by the social welfare weight $\psi^k(\theta')$. The choice rule

$$x_j^k(\boldsymbol{\xi},\theta') = \begin{cases} 1, & \text{if } u\big[c^k(j,\theta'),q_j^k(\theta');j,\theta'\big] + \xi_j^k - \frac{1}{\psi^k(\theta')}\Big[\Lambda_o c^k(j,\theta') + \Lambda_j q_j^k(\theta')\Big] \geq \\ & \quad \max_{j'}\Big\{ u\big[c^k(j',\theta'),q_{j'}^k(\theta');j',\theta'\big] + \xi_{j'}^k - \frac{1}{\psi^k(\theta')}\Big[\Lambda_o c^k(j',\theta') + \Lambda_{j'} q_{j'}^k(\theta')\Big]\Big\} \\ & \\ 0, & \text{otherwise} \end{cases} \tag{147}$$

And the key observation here is that this choice rule is exactly the same as that in the Pareto problem, and thus the complete markets allocation. Proposition 9 summarizes the result below.

Proposition 9 (Social Welfare Maximizing Allocations) *Let $\psi^k(\theta')$ be a vector of Social Welfare Weights. Define the* ***Social Welfare Function*** *as:*

$$\mathcal{W}_\psi = \sum_{\theta'}\int_k \psi^k(\theta')\int_{\boldsymbol{\xi}}\sum_j x_j^k(\boldsymbol{\xi},\theta')\Big\{u\big[c^k(\boldsymbol{\xi},\theta'),q_j^k(\boldsymbol{\xi},\theta');j,\theta'\big] + \xi_j^k\Big\}g(\boldsymbol{\xi},\theta')d\boldsymbol{\xi}\,di \tag{148}$$

Then a ***Social Welfare Maximizing allocation*** *is consumption allocations and commodity choice rules and $c_j^k(\boldsymbol{\xi},\theta')$, $q_j^k(\boldsymbol{\xi},\theta')$, $x_j^k(\boldsymbol{\xi},\theta')$ for all k,θ' to maximize $\mathcal{W}_\psi$ subject to resource constraints on all goods. The following conditions characterize the allocation:*

1. *For all k,θ', consumption allocations must satisfy:*

$$\psi^k(\theta')\,u_c\big[c^k(\theta'),q_j^k(\theta');j,\theta'\big] = \Lambda_o \quad \textit{and} \quad \psi^k(\theta')\,u_{q_j}\big[c^k(\theta'),q_j^k(\theta');j,\theta'\big] = \Lambda_j. \tag{149}$$

2. *For all k,θ' the commodity choice rule is*

$$x_j^k(\boldsymbol{\xi},\theta') = \begin{cases} 1, & \textit{if } u\big[c^k(j,\theta'),q_j^k(\theta');j,\theta'\big] + \xi_j^k - \frac{1}{\psi^k(\theta')}\Big[\Lambda_o c^k(j,\theta') + \Lambda_j q_j^k(\theta')\Big] \geq \\ & \quad \max_{j'}\Big\{ u\big[c^k(j',\theta'),q_{j'}^k(\theta');j',\theta'\big] + \xi_{j'}^k - \frac{1}{\psi^k(\theta')}\Big[\Lambda_o c^k(j',\theta') + \Lambda_{j'} q_{j'}^k(\theta')\Big]\Big\} \\ & \\ 0, & \textit{otherwise} \end{cases} \tag{150}$$

This allocation is a Pareto Efficient Allocation and coincides with a Complete Markets Allocation under some ex-ante transfers.

E. Appendix: An Alternative Formulation of the Objective Function

This section provides a representation of the objective function in (37) under the type 1 extreme value assumption that is (i) easy to solve using standard calculus, and (ii) provides a marginalist interpretation as to how a household or planner wants to allocate its resources across discrete choices.

Notation note: Without loss of generality, we simplify the presentation and do this only for one type of household.

We proceed in several steps and deep dive into derivations of results behind type 1 extreme value distributions. We used these results to establish the point that if the distribution $G(\boldsymbol{\xi})$ is of the type 1 extreme value form, and the choice rule is of the type:

$$x_j(\boldsymbol{\xi}) = \begin{cases} 1, & \text{if } V_j + \xi^i_j \;\; \geq \; \max_{j'} \left[\, V_{j'} + \xi^i_{j'} \, \right] \\ 0. & \end{cases} \tag{151}$$

where V_j is some object to be determined, then the choice probability ρ_j and expected value of ξ_j conditional on choosing j are:

$$\rho_j = \frac{\exp\left\{\frac{V_j}{\eta}\right\}}{\sum_k \exp\{\frac{V_{j'}}{\eta}\}} \quad , \quad \mathbb{E}\Big[\xi_j \Big| \text{Choose } j\Big] = -\eta \log \rho_j, \tag{152}$$

And then expected utility can be represented as

$$V^i = \sum_j \rho^i_j \Big[V^i_j - \eta \log \rho^i_j \Big]. \tag{153}$$

This formulation of expected utility is useful, because now then the problem of the household can be represented as chosen choice probabilities and consumption.

5.1. Choice probability

We first derive the choice probability given that the choice rule takes the form in (151). Recall that the Gumbel distribution has the following CDF and PDF:

$$G\left(\xi_j\right) = \exp\left\{-e^{-\xi_j/\eta}\right\}$$
$$g\left(\xi_j\right) = \frac{1}{\eta} \exp\left\{ - \left(\frac{\xi_j}{\eta} + e^{-\xi_j/\eta} \right) \right\}$$

Given the above form of $x_j(\boldsymbol{\xi})$, then the probability of choosing j, ρ_j is given by

$$\rho_j = \mathrm{P}\left\{V_j + \xi_j \geq V_{j'} + \xi_k \ , \ \forall j\right\}.$$

Then we compute this probability under the Gumbel distribution

$$\begin{aligned}
\rho_j &= \prod_{j' \neq j} P\left\{V_j + \xi_j \geq V_{j'} + \xi_{j'}\right\} \\
&= \prod_{j' \neq j} P\left\{\xi_{j'} \leq V_j - V_{j'} + \xi_j\right\} \\
&= \int \prod_{j' \neq j} P\left\{\xi_{j'} \leq V_j - V_{j'} + \xi_j \middle| \xi_j\right\} g\left(\xi_j\right) d\xi_j \\
\rho_j &= \int \prod_{j' \neq j} G\left(V_j - V_{j'} + \xi_j\right) g\left(\xi_j\right) d\xi_j.
\end{aligned}$$

Given the distribution and density of the Gumbel distribution

$$\rho_j = \int \prod_{j' \neq j} \exp\left\{-e^{-\left[V_j - V_{j'} + \xi_j\right]/\eta}\right\} \frac{1}{\eta} \exp\left\{-\left(\frac{\xi_j}{\eta} + e^{-\xi_j/\eta}\right)\right\} d\xi_j$$

Then impose a change of variables with $\widetilde{V} = V/\eta$ and $\widetilde{\xi}_j = \xi_j/\eta$, hence $d\xi_j = \eta d\widetilde{\xi}_j$:

$$\begin{aligned}
\rho_j &= \int \prod_{j' \neq j} \exp\left\{-e^{-\left[\widetilde{V}_j - \widetilde{V}_{j'} + \widetilde{\xi}_j\right]}\right\} \exp\left\{-\left(\widetilde{\xi}_j + e^{-\widetilde{\xi}_j}\right)\right\} d\widetilde{\xi}_j \\
\rho_j &= \int \prod_{j' \neq j} \exp\left\{-e^{-\widetilde{\xi}_j} \frac{\exp\left\{\widetilde{V}_{j'}\right\}}{\exp\left\{\widetilde{V}_j\right\}}\right\} \exp\left\{-\left(\widetilde{\xi}_j + e^{-\widetilde{\xi}_j}\right)\right\} d\widetilde{\xi}_j \\
\rho_j &= \int \exp\left\{-e^{-\widetilde{\xi}_j} \frac{\sum_{j' \neq j} \exp\left\{\widetilde{V}_{j'}\right\}}{\exp\left\{\widetilde{V}_j\right\}}\right\} \exp\left\{-\left(\widetilde{\xi}_j + e^{-\widetilde{\xi}_j}\right)\right\} d\widetilde{\xi}_j.
\end{aligned}$$

Then let $T = \sum_{j' \neq j} \exp\left\{\widetilde{V}_{j'}\right\} / \exp\left\{\widetilde{V}_j\right\}$:

$$\begin{aligned}
\rho_j &= \int \exp\left\{-e^{-\widetilde{\xi}_j} T\right\} \exp\left\{-\left(\widetilde{\xi}_j + e^{-\widetilde{\xi}_j}\right)\right\} d\xi_j \\
\rho_j &= \int \exp\left\{-e^{-\widetilde{\xi}_j}\left(T+1\right) - \widetilde{\xi}_j\right\} d\xi_j
\end{aligned}$$

One more change of variables: $y = e^{-\widetilde{\xi}_j}$, such that $dy = -e^{-\widetilde{\xi}_j} d\widetilde{\xi}_j$, hence $dy/y = -d\widetilde{\xi}_j$, and $\widetilde{\xi}_j = -\log y$:

$$\begin{aligned}
\rho_j &= \int \exp\left\{-y\left(T+1\right) + \log y\right\} \left(-\frac{dy}{y}\right) \\
\rho_j &= \int -\exp\left\{-y\left(T+1\right)\right\} dy \\
\rho_j &= \frac{1}{T+1} \\
\rho_j &= \frac{\exp\left\{\widetilde{V}_j\right\}}{\sum_{j'} \exp\left\{\widetilde{V}_{j'}\right\}} \\
\rho_j &= \frac{\exp\left\{V_j/\eta\right\}}{\sum_{j'} \exp\left\{V_{j'}/\eta\right\}}
\end{aligned}$$

Note that we use the observation that since $\widetilde{\xi}_j \in (-\infty, \infty)$, then $y \in [0, \infty)$.

5.2. Expected value of shock

We want $\mathbb{E}\left[\xi_j \middle| \text{Choose } j\right]$. The approach is to compute the CDF of the random variable "Given that j is chosen, ξ_j is less than x", and show that the distribution

$$F\left(x\right) = P\left(\xi_j \leq x \middle| \text{Choose } j\right)$$

is Gumbel, and hence apply results above to compute this conditional expectation directly. As a first step, by Baye's Law, this distribution can be expressed as

$$F\left(x\right) = \frac{P\left(\xi_j \leq x \text{ and Choose } j\right)}{P\left(\text{Choose } j\right)} = \frac{P\left(\xi_j \leq x \text{ and Choose } j\right)}{\rho_j\left(\theta\right)}$$

and then given our previous results, we just need to compute probability in the numerator. The numerator is

$$
\begin{aligned}
P\left(\xi_j \leq x \text{ and Choose } j\right) &= P\left(\xi_j \leq x \text{ and } V_j + \xi_j \geq V_{j'} + \xi_j \forall j' \neq j\right) \\
&= \int_{-\infty}^{\infty} \mathbf{1}\left[\xi_j \leq x\right] \prod_{j' \neq j} P\left(\xi_{j'} \leq V_j - V_{j'} + \xi_j\right) g\left(\xi_j\right) d\xi_j \\
&= \int_{-\infty}^{x} \prod_{j' \neq j} P\left(\xi_{j'} \leq V_j - V_{j'} + \xi_j\right) g\left(\xi_j\right) d\xi_j \\
&= \int_{-\infty}^{x} \prod_{j' \neq j} \exp\left\{-e^{-\left[V_j - V_{j'} + \xi_j\right]/\eta}\right\} \frac{1}{\eta} \exp\left\{-\frac{\xi_j}{\eta} - e^{-\frac{\xi_j}{\eta}}\right\} d\xi_j \\
&= \int_{-\infty}^{x/\eta} \prod_{j' \neq j} \exp\left\{-e^{-\left[\widetilde{V}_j - \widetilde{V}_{j'} + \widetilde{\xi}_j\right]}\right\} \exp\left\{-\widetilde{\xi}_j - e^{-\widetilde{\xi}_j}\right\} d\widetilde{\xi}_j \\
&= \int_{-\infty}^{x/\eta} \prod_{j' \neq j} \exp\left\{-e^{-\widetilde{\xi}_j} \frac{\exp\left\{\widetilde{V}_{j'}\right\}}{\exp\left\{\widetilde{V}_j\right\}}\right\} \exp\left\{-\widetilde{\xi}_j - e^{-\widetilde{\xi}_j}\right\} d\widetilde{\xi}_j \\
&= \int_{-\infty}^{x/\eta} \exp\left\{-e^{-\widetilde{\xi}_j} \frac{\sum_{j' \neq j} \exp\left\{\widetilde{V}_{j'}\right\}}{\exp\left\{\widetilde{V}_j\right\}}\right\} \exp\left\{-\widetilde{\xi}_j - e^{-\widetilde{\xi}_j}\right\} d\widetilde{\xi}_j \\
&= \int_{-\infty}^{x/\eta} \exp\left\{-e^{-\widetilde{\xi}_j} T\right\} \exp\left\{-\widetilde{\xi}_j - e^{-\widetilde{\xi}_j}\right\} d\widetilde{\xi}_j \\
P\left(\xi_j \leq x \text{ and Choose } j\right) &= \int_{-\infty}^{x/\eta} \exp\left\{-e^{-\widetilde{\xi}_j}\left(T+1\right) - \widetilde{\xi}_j\right\} d\widetilde{\xi}_j
\end{aligned}
$$

Then using the same change of variables as above and looking after the limits of integration, we have

$$
\begin{aligned}
P\left(\xi_j \leq x \text{ and Choose } j\right) &= \int_{\infty}^{e^{-x/\eta}} \exp\left\{-y\left(T+1\right) + \log y\right\} \left\{-\frac{1}{y} dy\right\} \\
&= \int_{e^{-x/\eta}}^{\infty} -\exp\left\{-y\left(T+1\right) + \log y\right\} \left\{-\frac{1}{y} dy\right\} \\
&= \int_{e^{-x/\eta}}^{\infty} \exp\left\{-y\left(T+1\right)\right\} dy \\
&= \left[-\frac{1}{T+1} \exp\left\{-y\left(T+1\right)\right\}\right]_{e^{-x/\eta}}^{\infty} \\
&= \frac{1}{T+1} \exp\left\{-e^{-x/\eta}\left(T+1\right)\right\} \\
&= \frac{1}{T+1} \exp\left\{-e^{-(x-\eta\log(T+1))/\eta}\right\} \\
P\left(\xi_j \leq x \text{ and Choose } j\right) &= \rho_j \exp\left\{-e^{-(x-\eta\log(T+1))/\eta}\right\}
\end{aligned}
$$

Combining this with the above,

$$F(x) = \frac{P(\xi_j \leq x \text{ and Choose } j)}{P(\text{Choose } j)} = \frac{P(\xi_j \leq x \text{ and Choose } j)}{\rho_j} = \exp\left\{-e^{-(x-\eta\log(T+1))/\eta}\right\}$$

Therefore the random variable x is also Gumbel distributed with mean

$$\mathbb{E}[x] = \eta \log(T+1)$$
$$\mathbb{E}[x] = \eta \log\left(\frac{1}{\rho_j}\right)$$
$$\mathbb{E}[x] = -\eta \log \rho_j$$

This is the expression given in the text.

5.3. Representation of Expected Utility

We can use the above to rewrite the expected utility of an individual *conditional on* the fact that the discrete choice follows the previous type of rule.

$$V^i(\theta) = \int_{\boldsymbol{\xi}} \sum_j x_j^i(\boldsymbol{\xi},\theta)\left\{u\left[c^i(\boldsymbol{\xi},\theta), q_j^i(\boldsymbol{\xi},\theta); j, \theta\right] + \xi_j^i\right\} g(\boldsymbol{\xi};\theta)\, d\boldsymbol{\xi}$$

Now here is an important issue. Note that in any problem we considered (incomplete, complete, planner), the first order condition for both consumption goods is always independent of $\boldsymbol{\xi}$. Using this observation expected utility is then

$$\int_{\boldsymbol{\xi}} \sum_j x_j^i(\boldsymbol{\xi},\theta)\left\{u\left[c^i(\boldsymbol{\xi},\theta), q_j^i(\boldsymbol{\xi},\theta); j, \theta\right] + \xi_j^i\right\} g(\boldsymbol{\xi};\theta)\, d\boldsymbol{\xi}$$

$$= \sum_j \rho_j^i(\theta)\left[u\left(c^i(j,\theta), q_j^i(\theta), j, \theta\right) + \mathbb{E}\left[\xi_j \middle| \text{Choose } j; \theta\right]\right]$$

$$= \sum_j \rho_j^i(\theta)\left[u\left(c^i(j,\theta), q_j^i(\theta), j, \theta\right) - \eta_\theta \log \rho_j^i(\theta)\right]$$

Where the last two lines use our arguments about the nature of the choice rule and then type 1 extreme value distribution. This then establishes Proposition 6.

5.4. Using Proposition 6 to Solve for Allocations

To further illustrate how this works, we illustrate how we can use the representation in Proposition 6 and (154) and study the problems we considered (incomplete, complete, planner). Perhaps not surprisingly, the allocations maximizing (154) lead to the same results in the body of the text.

Notation note. To add a bit more generality, we now allow for different types and the utility function only depends upon consumption.

Solving the incomplete markets problem. In incomplete markets, the objective function of the individual of type-θ is:

$$V = \max_{q_j(\theta),\rho_j(\theta)} \sum_j \left[\rho_j\left(\theta\right) u\left(c\left(j,\theta\right), q_j\left(\theta\right)\right) - \eta \rho_j\left(\theta\right) \log \rho_j\left(\theta\right)\right]$$

There is now a budget constraint for every choice:

$$c\left(j,\theta\right) + p_j q_j\left(\theta\right) \leq W(\theta) \quad \left[\lambda_j\left(\theta\right)\right] \quad \text{for each } j$$
$$\sum_j \rho_j\left(\theta\right) = 1 \quad \left[\chi\left(\theta\right)\right]$$

and now there is a new constraint that ensures the choice probabilities are probabilities. The first order conditions for $q_j\left(\theta\right)$ and $\rho_j\left(\theta\right)$ are:

$$q_j\left(\theta\right): \qquad 0 = \rho_j\left(\theta\right) u_q\left(c\left(j,\theta\right), q_j\left(\theta\right)\right) - \lambda_j\left(\theta\right) p_j$$
$$\rho_j\left(\theta\right): \qquad 0 = u\left(c\left(j,\theta\right), q_j\left(\theta\right)\right) - \eta \log \rho_j\left(\theta\right) - \eta - \chi\left(\theta\right)$$

With incomplete markets, the choices are not linked through the budget constraint. The choice probability can therefore be directly solved from the first order condition for $\rho_j\left(\theta\right)$.

$$\log \rho_j\left(\theta\right) + \frac{1}{\eta}\chi\left(\theta\right) = \frac{1}{\eta}\left[u\left(c\left(\theta\right), q_j\left(\theta\right)\right) - 1\right]$$

We can then exponentiate and sum over $\rho_j(\theta)$ to solve for the multiplier $\chi(\theta)$:

$$\begin{aligned}
\rho_j(\theta)\, e^{\frac{1}{\eta}\chi(\theta)} &= \exp\left\{\frac{1}{\eta}\left[u\left(c(\theta), q_j(\theta)\right) - 1\right]\right\} \\
e^{\frac{1}{\eta}\chi(\theta)} &= \sum_j \exp\left\{\frac{1}{\eta}\left[u\left(c(\theta), q_j(\theta)\right) - 1\right]\right\} \\
\frac{1}{\eta}\chi(\theta) &= \log\left[\sum_j \exp\left\{\frac{1}{\eta}\left[u\left(c(\theta), q_j(\theta)\right) - 1\right]\right\}\right] \\
\chi(\theta) &= \eta \log\left[\sum_j \exp\left\{\frac{1}{\eta}\left[u\left(c(\theta), q_j(\theta)\right) - 1\right]\right\}\right]
\end{aligned}$$

Substituting this back into the above

$$\begin{aligned}
\rho_j(\theta) &= \frac{\exp\left\{\frac{1}{\eta}\left[u\left(c(\theta), q_j(\theta)\right) - 1\right]\right\}}{\exp\left\{\frac{1}{\eta}\chi(\theta)\right\}} \\
\rho_j(\theta) &= \frac{\exp\left\{\frac{1}{\eta}\left[u\left(c(\theta), q_j(\theta)\right) - 1\right]\right\}}{\sum_j \exp\left\{\frac{1}{\eta}\left[u\left(c(\theta), q_j(\theta)\right) - 1\right]\right\}} \\
\rho_j(\theta) &= \frac{\exp\left\{\frac{1}{\eta}\left[u\left(c(\theta), q_j(\theta)\right)\right]\right\}}{\sum_j \exp\left\{\frac{1}{\eta}\left[u\left(c(\theta), q_j(\theta)\right)\right]\right\}}.
\end{aligned}$$

where the last line is the standard formula for the economy with incomplete markets.

Solving the planning problem. Now the social welfare function is:

$$\max_{q_j(\theta), \rho_j(\theta)} \sum_\theta \psi(\theta)\,\mu(\theta) \sum_j \left[\rho_j(\theta)\, u\left(c(j,\theta), q_j(\theta)\right) - \eta \rho_j(\theta) \log \rho_j(\theta)\right]$$

The resource constraints for each good are as follows, with an additional constraint on the choice probabilities:

$$\begin{aligned}
\sum_\theta \mu(\theta)\,\rho_j(\theta)\, q_j(\theta) &\le \sum_\theta \mu(\theta)\, y_j(\theta) \qquad [\Lambda_j] \quad \text{for each } j \\
\sum_\theta \mu(\theta) \sum_j \rho_j(\theta)\, c(j,\theta) &\le \sum_\theta \mu(\theta)\, y_o(\theta) \qquad [\Lambda_c] \\
\sum_j \rho_j(\theta) = 1 & \qquad [\mu(\theta)\,\chi(\theta)] \quad \text{for each } \theta
\end{aligned}$$

where recall that $\mu(\theta)$ is the mass of type θ agents. And note that to simplify the algebra, we normalize the multiplier on the final constraint by $\mu(\theta)$. The first order condition for $q_j(\theta)$ and $\rho_j(\theta)$

$$\begin{aligned}
c(j,\theta): &\qquad 0 = \psi(\theta)\,\rho_j(\theta)\,u_c(c(j,\theta), q_j(\theta)) - \rho_j(\theta)\langle\Lambda_c\rangle \\
q_j(\theta): &\qquad 0 = \psi(\theta)\,\rho_j(\theta)\,u_q(c(j,\theta), q_j(\theta)) - \rho_j(\theta)\langle\Lambda_j\rangle \\
\rho_j(\theta): &\qquad 0 = \psi(\theta)\left[u(c(j,\theta), q_j(\theta)) - \eta\log\rho_j(\theta) - \eta\right] - \left[\,\langle\Lambda_j\rangle q_j(\theta) + \langle\Lambda_c\rangle c(j,\theta)\,\right] - \chi(\theta)
\end{aligned}$$

From the first order condition for c and q_j we have

$$u_q(c(\theta), q_j(\theta)) = \frac{\Lambda_j}{\psi(\theta)}, \qquad u_c(c(j,\theta), q_j(\theta)) = \frac{\Lambda_c}{\psi(\theta)}$$

which are the same first order conditions associated with Proposition 5. Then working with the first order condition for $\rho_j(\theta)$ and following similar steps outlined for the incomplete markets problem we arrive at the following choice probabilities

$$\rho_j(\theta) = \frac{\exp\left\{\frac{1}{\eta}\left[u(c(j,\theta), q_j(\theta)) - \frac{1}{\psi(\theta)}\left[\,\langle\Lambda_j\rangle q_j(\theta) + \langle\Lambda_c\rangle c(j,\theta)\right]\right]\right\}}{\sum_j \exp\left\{\frac{1}{\eta}\left[u(c(j,\theta), q_j(\theta)) - \frac{1}{\psi(\theta)}\left[\,\langle\Lambda_j\rangle q_j(\theta) + \langle\Lambda_c\rangle c(j,\theta)\right]\right]\right\}}.$$

which is the same choice probability that the choice rule in Proposition 5 and equation (39) implies.

Solving the complete markets problem. To obtain the same choice probabilities under complete markets is then straight-forward. The objective function of the individual of type-θ is:

$$\max_{c(\theta), q_j(\theta), \rho_j(\theta)} \sum_j \left[\rho_j(\theta)\,u(c(j,\theta), q_j(\theta)) - \eta\rho_j(\theta)\log\rho_j(\theta)\right].$$

The consolidated budget constraint and probability constraint is

$$\sum_j \rho_j(\theta)\left[c(j,\theta) + p_j q_j(\theta)\right] \leq W(\theta) \qquad [\lambda(\theta)]$$

$$\sum_j \rho_j(\theta) = 1 \qquad [\chi(\theta)].$$

And the important piece here is that the individual understands that the price of an Arrow security associated with a choice $\varphi(\rho_j(\theta))$ is exactly the same as the choice probability $\rho_j(\theta)$ and hence we made this substitution into the consolidated budget constraint. The first order conditions for consumption and $\rho_j(\theta)$ are:

$$\begin{aligned} c(j,\theta): &\qquad 0 = \rho_j(\theta)\, u_c(c(j,\theta), q_j(\theta)) - \langle \lambda(\theta) \rangle \rho_j(\theta) \\ q_j(\theta): &\qquad 0 = \rho_j(\theta)\, u_q(c(j,\theta), q_j(\theta)) - \langle \lambda(\theta) \rangle p_j \rho_j(\theta) \\ \rho_j(\theta): &\qquad 0 = u(c(j,\theta), q_j(\theta)) - \eta \log \rho_j(\theta) - \eta - \langle \lambda(\theta) \rangle [c(j,\theta) + p_j q_j(\theta)] - \chi(\theta). \end{aligned}$$

Then notice how these conditions line up with the planner's first order conditions above. From the first order condition for c and q_j we have

$$u_q(c(\theta), q_j(\theta)) = p_j \lambda(\theta), \qquad u_c(c(j,\theta), q_j(\theta)) = \lambda(\theta),$$

and then again solving out for the choice probabilities we have

$$\rho_j(\theta) = \frac{\exp\left\{\frac{1}{\eta}\left[u(c(j,\theta), q_j(\theta)) - \langle \lambda(\theta) \rangle \left[c(j,\theta) + p_j q_j(\theta)\right]\right]\right\}}{\sum_j \exp\left\{\frac{1}{\eta}\left[u(c(j,\theta), q_j(\theta)) - \langle \lambda(\theta) \rangle \left[c(j,\theta) + p_j q_j(\theta)\right]\right]\right\}}.$$

which is the same choice probability in Proposition 2.

F. Appendix: A Spatial Economy

Our spatial environment is the following: Each choice j is a location in space. We remove the differentiated good and we assume the homogeneous good is produced by competitive firms, in all locations, and it is freely traded. The mass of each θ-type of households is normalized to one. Households begin "locationless" and the discrete choice nature of the problem is where they should live and work.[11]

Production in each location is

$$Y_j = F_j(\boldsymbol{\rho}_j), \tag{154}$$

where Y_j is output of the homogeneous good in location j, F_j is the production function that may depend upon characteristics of location j, and $\boldsymbol{\rho}_j = (\rho_j(\theta), \rho_j(\theta') \ldots)$ is the entire vector of all different type θ households working in location j. This last point means that output in location j may depend on the mix of type θ households working in that location. This production function could also exhibit an external productivity spillover.

[11] An alternative interpretation is to treat our θ-types as a households initial starting point, and thus the choice probability $\rho_j(\theta)$ characterizes the mass of migrants from location θ to location j.

Utility is

$$u\big[c^i(\boldsymbol{\xi},\theta)\big] + A_j(\boldsymbol{\rho}_j;\theta) + \xi^i_j, \tag{155}$$

where $A_j(\ldots,;\theta)$ is a j and θ specific amenity value. Similar to production, the dependence of A_j upon $\boldsymbol{\rho}_j$ means that the amenities in j, valued by type θ households depend upon the entire mix of households residing in that location. This function is treated as a spillover in the sense that individuals take this function as given and they do not internalize how their choices influence it.

Throughout this discussion, we employ a the type 1 extreme value distribution with shape parameter η_θ for each type. Given this assumption, we build on our streamlined presentation of the discrete choice problem in Section 5 and solve these problems by directly choosing choice probabilities.

Following our discussion above, we characterize the social planning problem using the type 1 extreme value distribution, and thus we can cast the problem in terms of choice probabilities. The planning problem is

$$\max_{q_j(\theta),\rho_j(\theta)} \sum_\theta \sum_j \rho_j(\theta) \Big\{ u\big[c(j,\theta)\big] + A_j(\boldsymbol{\rho}_j;\theta) - \eta_\theta \log \rho_j(\theta) \Big\} \tag{156}$$

$$\text{subject to:} \quad [\,\Lambda\,] \;\; \sum_j F_j(\boldsymbol{\rho}_j) \;\geq\; \sum_\theta \sum_j \rho_j(\theta) c(j,\theta) \tag{157}$$

$$[\,\Lambda_\theta\,] \;\; 1 \;=\; \sum_j \rho_j(\theta) \quad \forall \theta. \tag{158}$$

where the objective function is re-written now using our arguments in Section 5. The solution to this problem is characterized by the following first order conditions. First, there is the standard first order condition for consumption

$$u_c\big[c(j,\theta)\big] = \Lambda, \tag{159}$$

which as says the social welfare weighted marginal utility should be equated with the multiplier for location j. The first order condition for the choice probability becomes

$$V_j(\theta) - \eta_\theta + \Lambda \frac{\partial F_j}{\partial \rho_j(\theta)} + \psi(\theta) \sum_{\theta'} \frac{\partial A_j(\theta')}{\partial \rho_j(\theta)/\rho_j(\theta)} = \Lambda\, c(j,\theta) + \Lambda_\theta, \tag{160}$$

From (160) we can solve for the Social Planner's choice probability, $\rho_j^{SP}(\theta)$, which is:

$$\rho_j^{SP}(\theta) \propto \exp\left\{\eta_\theta^{-1}\left(A_j(\boldsymbol{\rho}_j;\theta) + u_c\frac{\partial F_j}{\partial \rho_j(\theta)} + \sum_{\theta'}\frac{\partial A_j(\theta')}{\partial \rho_j(\theta)/\rho_j(\theta)}\right)\right\}, \tag{161}$$

And notice that the consumption and utility terms don't show up because they don't vary with location since because the planner is equalizing marginal utility.

6.1. The Spatial Complete Markets Problem

The general spatial problem with complete markets is:

$$\max_{a^i(\boldsymbol{\xi},\theta),\ c^i(\boldsymbol{\xi},\theta),\ q^i_j(\boldsymbol{\xi},\theta),\ x^i_j(\boldsymbol{\xi},\theta)} \int_{\boldsymbol{\xi}} \sum_j x^i_j(\boldsymbol{\xi},\theta)\left\{u\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] + \xi^i_j\right\} g\big(\boldsymbol{\xi};\theta\big)\, d\boldsymbol{\xi}, \tag{162}$$

$$\text{subject to} \quad \sum_j x^i_j(\boldsymbol{\xi},\theta)\big[c^i(\boldsymbol{\xi},\theta) + p_j q^i_j(\boldsymbol{\xi},\theta) - w_j(\theta)\big] \leq\ + a^i(\boldsymbol{\xi},\theta) \quad \forall\, \boldsymbol{\xi}, \tag{163}$$

$$\int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta) a^i(\boldsymbol{\xi},\theta) d\boldsymbol{\xi} = 0. \tag{164}$$

where now for each shock, and hence choice, the household is facing a different endowment which is given by the wage rate $w_j(\theta)$. In this background, this would just reflect the private marginal product of labor that competitive labor markets would result in. Then from here, we can substitute all budget constraints (163) into the constraint for assets (164). The resulting problem is

$$\max_{a^i(\boldsymbol{\xi},\theta),\ c^i(\boldsymbol{\xi},\theta),\ q^i_j(\boldsymbol{\xi},\theta),\ x^i_j(\boldsymbol{\xi},\theta)} \int_{\boldsymbol{\xi}} \sum_j x^i_j(\boldsymbol{\xi},\theta)\left\{u\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] + \xi^i_j\right\} g\big(\boldsymbol{\xi};\theta\big)\, d\boldsymbol{\xi}, \tag{165}$$

$$\text{subject to} \quad [\,\lambda^i(\theta)\,]: \quad \int_{\boldsymbol{\xi}} \varphi(\boldsymbol{\xi},\theta)\left\{\sum_j x^i_j(\boldsymbol{\xi},\theta)\big[c^i(\boldsymbol{\xi},\theta) + p_j q^i_j(\boldsymbol{\xi},\theta) - w_j(\theta)\big]\right\} d\boldsymbol{\xi} = 0. \tag{166}$$

Then this problem gives rise to the two first order conditions for consumption

$$u_{q_j}\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] = \lambda^i(\theta) p_j. \tag{167}$$

Then similarly for the non-differentiated good

$$u_c\big[c^i(\boldsymbol{\xi},\theta), q^i_j(\boldsymbol{\xi},\theta); j,\theta\big] = \lambda^i(\theta) \tag{168}$$

and then the following first order condition for the choice probability

$$V_j(\theta) - \eta_\theta + \lambda^i(\theta)w_j(\theta) - \lambda(\theta)\big[c(\theta) + p_j q_j(\theta)\big] = 0, \tag{169}$$

where we make the usual arguments to drop the dependence upon the shock and the identity i.

Now from here, let's specialize this by removing the differentiated good. The first order condition for the choice probability becomes

$$V_j(\theta) - \eta_\theta + \lambda(\theta)w_j(\theta) - \lambda(\theta)c(j,\theta) = 0. \tag{170}$$

Here one can see the key distinction between the role that incomplete insurance vs. spillovers are playing. In the complete markets allocation, first order condition needs to balance out the private gain from being in a location which is given by the wage rate $w_j(\theta)$ versus the private cost of being in that location in terms of expenditure. This has the same flavor as in (161), but the key distinction is that this is private, not social. And this distinction comes about because of spillovers either through production or amenities. From (170) we can solve for the choice probability which is

$$\rho_j^{CM}(\theta) \propto \exp\left\{\eta_\theta^{-1}\bigg(A_j(\boldsymbol{\rho}_j;\theta) + u_c(\theta)w_j(\theta)\bigg)\right\}. \tag{171}$$

with the last two terms reflecting the private benefit of being in j versus the private consumption cost of being in j dictating the location choice.

No Spillovers; Constant Marginal Product of Labor. Here we turn off the spillovers (both productivity and amenity). Here we do so by considering linear production technologies with TFP levels Z_j and hence the marginal product of labor is constant. In this case the optimal choice probability of a Social Planner facing No Spillovers collapses to:

$$\rho_j^{SP-NS}(\theta) \propto \exp\left\{\eta_\theta^{-1}\bigg(u_c\, Z_j\bigg)\right\}. \tag{172}$$

This says that the planner locates households based on each location's productivity, weighted by the (common) marginal utility of consumption.