Example Model Solved with Tools Presented in Quantitative Macro Lecture

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1 Introduction

- Below I write-down and then compute a simple model common in the firm dynamics literature, a general equilibrium model of firm capital adjustment.
- The model itself is a little more complicated than it needs to be so if only interested in the computation you can skip to section 3.

2 Model

- This is a simple general equilibrium model which represents a version of the stationary steady state of the model found in Khan and Thomas (2005)
- The firm's states are (k, z), capital and productivity
- The **firm** problem is to maximize the expected present discounted value of dividends $V(k, z)$, which can be stated recursively

$$V(k,z) = \max_{k',n} zf(n,k) - [k' - (1-\delta)k] - AC(k,k') - wn + \mathbb{E}\left[mV(k',z')\right]$$

where m is the household stochastic discount factor, w is the wage, $AC(k, k')$ is an adjustment cost the firm must pay to change its capital stock, $f(n, k)$ is a decreasing returns to scale production technology.

- The **household** problem is

$$\begin{aligned} W(\lambda) &= \max_{c,n,\lambda'} U(c, 1-n^h) + \beta W(\lambda') \\ &\text{s.t.} \\ c + \int_{\mathbf{S}} \rho_1(k',z')d\lambda'(k',z') &\leq wn + \int \rho_0(k,z)d\lambda(k,z) \end{aligned}$$

where ρ_1 is the price of new shares, ρ_0 is the price of old shares, λ is the portfolio of firms held by the household with λ' chosen for next period, w is the wage

- The household optimality conditions (under $c = C$ and $n = N$) are

1. Labor supply

$$w = \frac{U_2(C, 1-N)}{U_1(C, 1-N)}$$

2. Stochastic discount factor (yes, this is β in steady state but we don't need to impose steady state at this point)

$$m = \beta \frac{U_1(C', 1-N')}{U_1(C, 1-N)}$$

- We can now re-write the firm's problem under these equilibrium conditions.
- Substituting m into the firm's value function

$$\begin{aligned} V(k,z) &= \max_{k',n} zf(n,k) - k' - (1-\delta)k - wn - AC(k,k') + \beta \left[\frac{U_1(C', 1-N')}{U_1(C, 1-N)} V(k', z') \right] \\ U_1(C, 1-N) V(k,z) &= \max_{k',n} U_1(C, 1-N) \left[zf(n,k) - k' - (1-\delta)k - wn \right] + \beta \left[U_1(C', 1-N')) V(k', z') \right] \end{aligned}$$

- Letting $p = U_1(C, 1-N)$, and defining $v(k,z) = pV(k,z)$ we have the following problem

$$v(k,z) = \max_{k',n} p \left[zf(n,k) - k' - (1-\delta)k - AC(k,k') - wn \right] + \beta \mathbb{E} \left[v(k', z', s', \mu') \right]$$

- Given a Cobb-Douglas functional for the production technology $f(n,k) = k^\alpha n^\nu$, where $\alpha + \nu < 1$, we obtain $n(k,z,w)$ from the $FOC(n)$

$$n(k,z,w) = \left[\frac{\nu z k^\alpha}{w} \right]^{\frac{1}{1-\nu}}$$

- This can be used to determine period profits

$$\pi(k,z,w) = zf(n(k,w,z),k) - wn(k,w,z)$$

- Substituting this into the firm's problem we have the Bellman equation

$$v(k,z) = \max_{k',n} p \left[\pi(k,z,w) - k' - (1-\delta)k - AC(k,k') \right] + \beta \mathbb{E} \left[v(k', z') \right]$$

- Now consider the following functional form for preferences $U(c, 1-n) = \log c - \psi n$, in this case

$$\begin{aligned} p &= U_1(C, 1-N) = \frac{1}{C} \\ w &= \frac{U_2(C, 1-N)}{U_1(C, 1-N)} = C\psi = \frac{\psi}{1/C} = \frac{\psi}{p} \end{aligned}$$

- **Equilibrium** - An equilibrium requires that the goods market clears

$$\begin{aligned} C &= Y - I \\ C &= \int zf(n,k) d\lambda(n,k) - \int \left[k'(k,z) - (1-\delta)k \right] + AC(k, k'(k,z)) \, d\lambda(k,z) \end{aligned}$$

and that prices are consistent

$$
\begin{aligned}
p &= 1/C \\
w &= \psi/p
\end{aligned}
$$

3 Algorithm

- We assume that z takes on values in a discrete set $\mathcal{Z}$ and evolves stochastically according to a transition matrix P which approximates an AR(1) process
- We proceed as in the lecture. The equilibrium price is p. From that we can determine $w = \psi/p$. We then solve the firm's problem given (w, p), compute the stationary distribution, aggregate to compute $C(p)$ and check whether $C(p) = 1/p$. We then bisect using the fact that if $C(p) > 1/p$ then their is excessive production and too little investment, which implies we should *increase* p.

3.1 Collocation

- The Bellman equation of interest is

$$
v(k, z) = \max_{k'} p\left[\pi(k, z, w) - k' - (1-\delta)k - AC(k, k')\right] + \beta\mathbb{E}\left[v(k', z')\right]
$$

- We introduce the notation $s = (k, z)$, write the flow pay-off as a function $F(s, k', p)$, which includes the optimal choice of labor and the equilibrium wage of $w = \psi/p$.

$$
v(s) = \max_{s} F(s, k', p) + \beta\mathbb{E}\left[v(k', z')\right]
$$

- In this case I'm actually going to do something a little different to what we saw in class and instead just solve for $v^2(k, z) = \mathbb{E}\left[v(k, z')\right]$
- Writing down the Bellman equation for v^2 I have

$$
v^2(k, z) = \sum_{z'} P(z, z')\left\{\max_{s} F(s, k', p) + \beta v^2(k', z')\right\}
$$

- Why would I do this? Given that the adjustment costs are zero when the firm adjusts downwards, there may be a kink in v^1, therefore if it's simple to solve just for v^2 (which potentially smooths out this kink) then why not? I also keep this property of only computing expectations once
- Substituting in splines and stacking equations

$$
\Phi(s)c = (\mathbf{P} \otimes \mathbf{I}_{J/K})\left[\max_{k'} F(s, k', p) + \beta\Phi([k', s_2])c\right]
$$

- The Jacobian is (line 15 of solve_valfunc.m)

$$
jac = \Phi(s) - \beta(\mathbf{P} \otimes \mathbf{I}_{J/K})\Phi([k'(s), s_2])
$$

- In the code I refer to $[\max_{k'} F(s,k',p) + \beta\Phi([k',s_2])c]$ as v^1 although this is just a place holder and I compute $v^2 = (\mathbf{P} \otimes \mathbf{I}_{J/K})v^1$ immediately after (line 12 of solve_valfunc.m)