Appendix II - Not for Publication

Firm and Worker Dynamics in a Frictional Labor Market

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This Appendix contains the proof of the joint surplus representation in the fully dynamic model, the main result of Section 3 in the main text, and computational details. Section A lays out the notation for the fully dynamic model. Section B develops the proof. Section C contains details on the computation of the stationary equilibrium of the model.

A Notation for dynamic model

We first specify the value function of an individual worker i in a firm with arbitrary state x: $V(x,i)$. We then specify the value function of the firm: $J(x)$. Combining all workers' value functions with that of the firm we define the joint value: $\Omega(x)$. We then apply the assumptions from Section 2 which allow us to reduce (x) to only the number of workers and productivity of the firm, (z,n). Finally we take the continuous work force limit to derive a Hamilton-Jacobi-Bellman (HJB) equation for $\Omega(z,n)$ Applying the definition of total surplus used above, we obtain a HJB equation in $S(z,n)$ which we use to construct the equilibrium.

A.1 Worker value function: V

Let U be the value of unemployment. Define separately worker i's value when employed at firm x *before* the quit, layoff and exit decisions, $\boldsymbol{V}(x,i)$, and their value *after* these decisions, $V(x,i)$.

Value of unemployment. Let $h_U(x)$ denote how the state of firm x is updated when it hires an unemployed worker.[1] Let $\mathcal{A}$ denote the set of firms making job offers that an unemployed worker would accept. The value of unemployment U therefore satisfies

$$\rho U = b + \lambda^U(\theta) \int_{x \in \mathcal{A}} \left[\boldsymbol{V}\left(h_U(x), i\right) - U\right] dH_v(x)$$

where H_v is the vacancy-weighted distribution of firms. If $x \notin \mathcal{A}$, then the worker remains unemployed.

Stage I. To relate the value of the worker pre separation, $\boldsymbol{V}(x,i)$, to that post separation, $V(x,i)$, we require the following notation regarding firm and co-worker actions. Since workers do not form 'unions' within the firm, all of these actions are taken as given by worker i.

- Let $\epsilon(x) \in \{0,1\}$ denote the exit decision of firm, and $\mathcal{E} = \{x : \epsilon(x) = 1\}$ the set of x's for which the firm exits.
- Let $\ell(x) \in \{0,1\}^{n(x)}$ be a vector of zeros and ones of length $n(x)$, with generic entry $\ell_i(x)$, that characterizes the firm's decision to lay off incumbent worker $i \in \{1,\ldots,n(x)\}$, and $\mathcal{L} = \{(x,i) : \ell_i(x) = 1\}$ the set of (x,i) such that worker (x,i) is laid off.
- Let $q^U(x) \in \{0,1\}^{n(x)}$ be a vector of length $n(x)$, with generic entry $q_i^U(x)$ that characterizes an incumbent workers' decisions to quit, and $\mathcal{Q}^U = \left\{(x,i) : q_i^U(x) = 1\right\}$ the set of (x,i) such that worker (x,i) quits into unemployment.

[1] For example, size would be update from n to $n+1$ and possibly some of the incumbent wages would be bargained down.

- Let $\kappa(x) = (1 - \boldsymbol{\ell}(x)) \circ (1 - q_U(x))$ be an element-wise product vector that identifies workers that are kept in the firm, and $\mathcal{S} = \mathcal{L} \cup \mathcal{Q}^U = \{(x,i) : \kappa_i(x) = 0\}$, the set of (x,i) such that worker (x,i) separates into unemployment.
- Let $s(x, \kappa(x))$ denote how the state of firm x is updated when workers identified by $\kappa(x)$ are kept. This includes any renegotiation.

Given these sets and functions, the pre separation value $\boldsymbol{V}(x,i)$ satisfies:

$$\boldsymbol{V}(x,i) = \underbrace{\epsilon(x)U}_{\text{Exit}} + (1-\epsilon(x))\Big[\underbrace{\mathbb{I}_{\{(x,i)\notin\mathcal{S}\}} V(s(x,\kappa(x)),i)}_{\text{Continuing employment}} + \underbrace{\mathbb{I}_{\{(x,i)\in\mathcal{S}\}} U}_{\text{Separations and Quits}}\Big]$$

Stage II. It is helpful to characterize the continuation value of employment post separation decisions, $V(x,i)$, in terms of the three distinct types of events described in Figure . First, the value changes due to *'Direct'* labor markets shocks to worker i, $V_D(x,i)$. These include her match being destroyed exogenously or meeting a new potential employer. Second, the value changes due to labor market shocks hitting other workers in the firm, $V_I(x,i)$, including their matches being exogenously destroyed or them meeting new potential employers. These events have an *'Indirect'* impact on worker i. Third, the value changes due to events on the *'Firm'* side, $V_F(x,i)$, including the firm contacting new workers and receiving productivity shocks. Combining events and exploiting the fact that in continuous time they are mutually exclusive, we obtain the following, where $w(x,i)$ is the wage paid to worker i:

$$\rho V(x,i) = w(x,i) + \rho V_D(x,i) + \rho V_I(x,i) + \rho V_F(x,i).$$

The wage function $w(x,i)$ includes the transfers between worker i and the firm that may occur at the stage of vacancy posting (after separations and before the labor market opens), as discussed in the Appendix Section A in the context of the static example. These transfers can depend on the entire wage distribution inside the firm which is subsumed in the state vector x.

Direct events. We first characterize changes in value due to labor market shocks directly to worker i in firm x, $V_D(x,i)$. Exogenous separation shocks arrive at rate δ and draws of outside offers arrive at rate $\lambda^E(\theta)$ from the vacancy-weighted distribution of firms H_v. If worker i receives a sufficiently good outside offer from x', she quits to the new firm. We denote by $\mathcal{Q}^E(x,i)$ the set of such quit-firms x' for i. Otherwise, the worker remains with the current firm but with an updated contract. Therefore $V_D(x,i)$ satisfies

$$\begin{aligned}\rho V_D(x,i) = \underbrace{\delta\left[U - V(x,i)\right]}_{\text{Exogenous separation}} &+ \lambda^E(\theta)\underbrace{\int_{x'\in\mathcal{Q}^E(x,i)}\left[\boldsymbol{V}\left(h_E(x,i,x'),i\right) - V(x,i)\right]dH_v(x')}_{EE\ \text{Quit}} \\ &+ \lambda^E(\theta)\underbrace{\int_{x'\notin\mathcal{Q}^E(x,i)}\left[\boldsymbol{V}\left(r(x,i,x'),i\right) - V(x,i)\right]dH_v(x')}_{\text{Retention}},\end{aligned}$$

where $h_E(x,i,x')$ describes how the state of a poaching firm x' gets updated when it hires worker i from firm x. Similarly, $r(x,i,x')$ updates x when—after meeting firm x'—worker i in firm x is retained and renegotiates its value. In all functions with three arguments (x,i,x'), the first argument denotes the origin firm, the second identifies the worker, and the third the potential destination firm.

Indirect events. We next characterize changes in value due to the same labor market shocks hitting other workers in firm x, $V_I(x,i)$. The value $V_I(x,i)$ satisfies

$$\rho V_I(x,i) = \sum_{j\neq i}^{n(x)} \Bigg\{ \underbrace{\delta\left[\boldsymbol{V}(d(x,j),i) - V(x,i)\right]}_{\text{Exogenous separation}} + \lambda^E(\theta) \underbrace{\int_{x'\in\mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(q_E(x,j,x'),i\right) - V(x,i)\right] dH_v(x')}_{EE\text{ Quit}}$$
$$+ \lambda^E(\theta) \underbrace{\int_{x'\notin\mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(r(x,j,x'),i\right) - V(x,i)\right] dH_v(x')}_{\text{Retention}} \Bigg\},$$

where $d(x,j)$ updates x when worker j exogenously separates, and $q_E(x,j,x')$ when worker j quits to firm x'.

Firm events. Finally, we characterize changes in value due to events that directly impact the firm and hence indirectly its workers, $V_F(x,i)$. Taking as given the firm's vacancy posting policy $v(x)$ and other actions, $V_F(x,i)$ satisfies

$$\begin{aligned}
\rho V_F(x,i) \quad &= \\
UE\text{ Hire} \quad & \phi q(\theta) v(x) \left[\boldsymbol{V}(h_U(x),i) - V(x,i)\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\text{ Threat} \quad & +\phi q(\theta) v(x) \left[\boldsymbol{V}(t_U(x),i) - V(x,i)\right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\text{ Hire} \quad & +(1-\phi)\, q(\theta) v(x) \int_{x\in\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(h_E(x',i',x),i\right) - V(x,i)\right] dH_n(x',i') \\
EE\text{ Threat} \quad & +(1-\phi)\, q(\theta) v(x) \int_{x\notin\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(t_E(x',i',x),i\right) - V(x,i)\right] dH_n(x',i') \\
\text{Shock} \quad & +\Gamma_z\left[\boldsymbol{V},V\right](x,i)
\end{aligned}$$

where $t_U(x)$ updates x when an unemployed worker is met and not hired, but could be possibly used as a threat in firm x. Similarly, $t_E(x',i',x)$ updates x when worker i' employed at firm x' is met, not hired, but could be used as a threat. And, with a slight abuse of notation, $H_n(x',i')$ gives the joint distribution of firms x' and worker types within firms i'.

Finally, $\Gamma_z\left[\boldsymbol{V},V\right](x,i)$ identifies the contribution of productivity shocks z to the Bellman equation. At this stage we only require that the productivity process is Markovian with an infinitesimal generator. Later we will specialize this to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$ such that

$$\begin{aligned}
\Gamma_z\left[\boldsymbol{V},V\right](x,i) \quad &= \quad \mu(z) \lim_{dz\to 0} \frac{\boldsymbol{V}((x,z+dz),i) - V(x,z,i)}{dz} \\
&+ \quad \frac{\sigma^2(z)}{2} \lim_{dz\to 0} \frac{\boldsymbol{V}((x,z+dz),i) + \boldsymbol{V}((x,z-dz),i) - 2V(x,z,i)}{dz^2}
\end{aligned} \tag{1}$$

In the case that $\boldsymbol{V} = V$, this becomes the standard expression for a diffusion featuring the first and second derivatives of V with respect to z: $\Gamma_z[V](x,i) = \mu(z)V_z(x,z,i) + \frac{1}{2}\sigma(z)^2 V_{zz}(x,z,i)$.[2]

In the event productivity changes or $n(x)$ changes because of exogenous labor market events, the worker will want to reassess whether to stay with the firm or not. Additionally, the firm may want to reassess whether to exit or fire some workers. Bold values $\boldsymbol{V}$ capture any case where the state changes.

[2]Note that in (1) we abuse notation and write the state as (x,z) with some redundancy since z is clearly a member of x. We also note that we are not constrained to a diffusion process. We could also consider a Poisson process where, at exogenous rate η, z jumps according to the transition density $\Pi(z,z')$: $\Gamma_z\left[\boldsymbol{V},V\right](x,i) = \eta\left[\sum_{z'\in Z} \boldsymbol{V}((x,z'),i)\,\Pi(z',z) - V(x,z,i)\right]$.

A.2 Firm value function: J

Consistent with the notation we used for workers' values, let $\boldsymbol{J}(x)$ and $J(x)$ be the values of the firm at the corresponding points of an interval dt. For now, we take the vacancy creation decision $v(x)$ as given. At the end of the section we describe the expected value of an entrant firm.

Stage I. Consistent with the first stage worker value function, we define the firm value before the exit/layoff/quit decision, where we recall that ϑ is the firm's value of exit, or scrap value:

$$\boldsymbol{J}(x) = \epsilon(x)\vartheta + [1-\epsilon(x)]\, J(s(x,\kappa(x))).$$

Stage II. Given a vacancy policy $v(x)$, let $J(x)$ be the value of a firm with state x *after* the layoff/quit, exit. It is convenient to split the value of the firm, as we did for the worker, into three components

$$\rho J(x) = \underbrace{y(x) - \sum_{i=1}^{n(x)} w_i(x,i)}_{\text{Flow profits}} \quad + \quad \underbrace{\rho J_W(x)}_{\text{Workforce events}} \quad + \quad \underbrace{\rho J_F(x) - c(v(x),x)}_{\text{Firm events net of vacancy costs}} \quad .$$

For a given policy $v(x)$ there is a set of associated transfers between workers and the firm which, as for the worker value function, are implicit in the wage function $w(x,i)$.

The component $J_W(x)$ is given by

$$\begin{aligned}
&\rho J_W(x) \quad = \\
&\text{Destruction} \qquad \delta \sum_{i=1}^{n(x)} \left[\boldsymbol{J}(d(x,i)) - J(x)\right] \\
&EE \text{ Quit} \qquad + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) - J(x)\right] dH_v\left(x'\right) \\
&\text{Retention} \qquad + \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(r\left(x,i,x'\right)\right) - J(x)\right] dH_v\left(x'\right).
\end{aligned}$$

The component $J_F(x)$ is given by

$$\begin{aligned}
&\rho J_F(x) \quad = \\
&UE \text{ Hire} \qquad \phi q(\theta) v(x) \left[\boldsymbol{J}(h_U(x)) - J(x)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
&UE \text{ Threat} \qquad + \phi q(\theta) v(x) \left[\boldsymbol{J}(t_U(x)) - J(x)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
&EE \text{ Hire} \qquad + (1-\phi)\, q(\theta) v(x) \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) - J(x)\right] dH_n\left(x',i'\right) \\
&EE \text{ Threat} \qquad + (1-\phi)\, q(\theta) v(x) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(t_E\left(x',i',x\right)\right) - J(x)\right] dH_n\left(x',i'\right) \\
&\text{Shock} \qquad + \Gamma_z[\boldsymbol{J}, J](x)
\end{aligned}$$

Recall that, in continuous time at most one contact is made per instant. Either one worker is exogenously separated, or one worker is contacted by another firm, or one worker is met by posting vacancies (at rate $q(\theta)v(x)$), or a shock hits the firm. We have bold $\boldsymbol{J}$'s in each line since after any of these events, the firm may want to layoff some workers or exit, and workers may want to quit.

Entry. The expected value of an entrant firm is

$$J_0 = -c_0 + \int \boldsymbol{J}\left(x_0\right) d\Pi_0\left(z_0\right) \tag{2}$$

where x_0 is the state of the entrant firm which includes only the random productivity value z_0 drawn from Π_0 since we assumed the initial number of workers is 0. The argument of the integral is $\boldsymbol{J}$, which incorporates the firm's decision to exit or operate after observing z_0. Entry occurs when $J_0 > 0$.

B Derivation of the joint value function Ω

We define the **joint value** of the firm and its employed workers $\Omega\left(x\right) := J\left(x\right) + \sum_{i=1}^{n(x)} V\left(x, i\right)$. We also define the joint value before exit/quit/layoff decisions: $\boldsymbol{\Omega}\left(x\right) := \boldsymbol{J}\left(x\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(x, i\right)$.

B.1 Combinining worker and firm values

In this section, we show that summing firm and worker values, then applying these definitions delivers the following Bellman equation for the joint value:

$$\begin{aligned}
\rho\Omega\left(x\right) \quad &= \quad y\left(x\right) - c\left(v\left(x\right), x\right) \\
\text{Destruction} \quad &+ \sum_{i=1}^{n(x)} \delta\left[\boldsymbol{\Omega}\left(d(x,i)\right) + U - \Omega\left(x\right)\right] \\
\text{Retention} \quad &+ \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(r\left(x, i, x'\right)\right) - \Omega\left(x\right)\right] dH_v\left(x'\right) \\
EE \text{ Quit} \quad &+ \lambda^E(\theta) \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(q_E\left(x, i, x'\right)\right) + \boldsymbol{V}\left(h_E\left(x, i, x'\right), i\right) - \Omega\left(x\right)\right] dH_v\left(x'\right) \\
UE \text{ Hire} \quad &+ \phi q(\theta) v\left(x\right)\left[\boldsymbol{\Omega}\left(h_U\left(x\right)\right) - U - \Omega\left(x\right)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE \text{ Threat} \quad &+ \phi q(\theta) v\left(x\right)\left[\boldsymbol{\Omega}\left(t_U\left(x\right)\right) - \Omega\left(x\right)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE \text{ Hire} \quad &+ (1-\phi) q(\theta) v\left(x\right) \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(h_E\left(x', i', x\right)\right) - \boldsymbol{V}\left(h_E\left(x', i', x\right), i'\right) - \Omega\left(x\right)\right] dH_n\left(x', i'\right) \\
EE \text{ Threat} \quad &+ (1-\phi) q(\theta) v\left(x\right) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(t_E\left(x', i', x\right)\right) - \Omega\left(x\right)\right] dH_n\left(x', i'\right) \\
\text{Shock} \quad &+ \Gamma_z\left[\boldsymbol{\Omega}, \Omega\right]\left(x\right).
\end{aligned} \tag{3}$$

This joint value is only written in terms of other joint values and worker values. However, it involves both firm and worker decisions through the sets $\mathcal{A}$, $\mathcal{Q}^E$ and the vacancy policy, $v(x)$.

Derivation. We start by computing the sum of the workers' values at a particular firm. Summing values of all the employed workers

$$
\begin{aligned}
\rho \sum_{i=1}^{n(x)} V(x,i) &= \sum_{i=1}^{n(x)} w(x,i) \\
\text{Destructions} \quad & + \sum_{i=1}^{n(x)} \delta \left[U - V(x,i) \right] \\
\text{Retentions} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{V}\left(r\left(x,i,x'\right), i\right) - V(x,i) \right] dH_v\left(x'\right) \\
EE \text{ Quits} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{V}\left(h_E\left(x,i,x'\right)\right) - V(x,i) \right] dH_v\left(x'\right) \\
\text{Incumbents} \quad & + \sum_{i=1}^{n(x)} \rho V_I(x,i) \\
\text{Firm} \quad & + \sum_{i=1}^{n(x)} \rho V_D(x,i)
\end{aligned}
$$

where the indirect terms due to incumbents and the firm can be written as:

$$
\begin{aligned}
\sum_{i=1}^{n(x)} \rho V_I(x,i) &= \\
\text{Destructions} \quad & \sum_{i=1}^{n(x)} \sum_{j \neq i}^{n(x)} \delta \left[\boldsymbol{V}\left(d(x,j), i\right) - V(x,i) \right] \\
\text{Retentions} \quad & + \sum_{i=1}^{n(x)} \sum_{j \neq i}^{n(x)} \lambda^E \int_{x' \notin \mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(r\left(x,j,x'\right), i\right) - V(x,i) \right] dH_v\left(x'\right) \\
EE \text{ Quits} \quad & + \sum_{i=1}^{n(x)} \sum_{j \neq i}^{n(x)} \lambda^E \int_{x' \in \mathcal{Q}^E(x,j)} \left[\boldsymbol{V}\left(q_E\left(x,j,x'\right), i\right) - V(x,i) \right] dH_v\left(x'\right) \ , \\
\sum_{i=1}^{n(x)} \rho V_F(x,i) &= \\
UE \text{ Hires} \quad & qv(x)\,\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(h_U(x), i\right) - V(x,i) \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE \text{ Threats} \quad & + qv(x)\,\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(t_U(x), i\right) - V(x,i) \right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE \text{ Hires} \quad & + qv(x)\,(1-\phi) \sum_{i=1}^{n(x)} \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(h_E\left(x',i',x\right), i\right) - V(x,i) \right] dH_n\left(x',i'\right) \\
EE \text{ Threats} \quad & + qv(x)\,(1-\phi) \sum_{i=1}^{n(x)} \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(t_E\left(x',i',x\right), i\right) - V(x,i) \right] dH_n\left(x',i'\right) \\
\text{Shocks} \quad & + \sum_{i=1}^{n(x)} \Gamma_z[\boldsymbol{V}, V](x,i)
\end{aligned}
$$

We now collect terms.

Destructions. When worker i separates from firm x, the sum of the changes in values of all employed workers at its own firm is given by:

$$\text{Destructions} \quad = \quad \delta\left[U - V\left(x,i\right)\right] + \delta \sum_{j\neq i}^{n(x)} \left[\boldsymbol{V}\left(d(x,i),j\right) - V\left(x,j\right)\right] = \delta\left[U + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(d(x,i),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right]$$

Retentions. When i renegotiates at firm x, the sum of the changes in values of all employed workers at its own firm is given by:

$$\begin{aligned}
\text{Retentions} \quad &= \quad \lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r\left(x,i,x'\right),i\right) - V\left(x,i\right)\right] dH_v\left(x'\right) + +\lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \sum_{j\neq i}^{n(x)} \left[\mathbf{V}\left(r\left(x,i,x'\right),j\right) - V\left(x,j\right)\right] dH_v\left(x'\right) \\
&= \quad \lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\mathbf{V}\left(r\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dH_v\left(x'\right) \\
&= \quad \lambda^E \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\sum_{j=1}^{n(x)} \mathbf{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dH_v\left(x'\right)
\end{aligned}$$

Quits. Similarly, when i quits firm x, the sum of the changes in values of all employed workers at its own firm is given by:

$$EE\text{ Quits} = \lambda^E \int_{x'\in\mathcal{Q}(x,i)} \left[\boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dH_v\left(x'\right)$$

Combining terms. Before summing up all these terms, define for convenience the total worker value:

$$\begin{aligned}
\rho\overline{V}\left(x\right) \quad &= \quad \sum_{i=1}^{n(x)} w\left(x,i\right) \\
\text{Destructions} \quad & \quad + \sum_{i=1}^{n(x)} \delta\left[U + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(d(x,i),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] \\
\text{Retentions} \quad & \quad +\lambda^E \sum_{i=1}^{n(x)} \int_{x'\notin\mathcal{Q}^E(x,i)} \left[\sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dH_v\left(x'\right) \\
EE\text{ Quits} \quad & \quad +\lambda^E \sum_{i=1}^{n(x)} \int_{x'\in\mathcal{Q}^E(x,i)} \left[\boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \sum_{j=1}^{n(x)} V\left(x,j\right)\right] dH_v\left(x'\right) \\
UE\text{ Hires} \quad & \quad +qv\left(x\right)\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(h_U\left(x\right),i\right) - V\left(x,i\right)\right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE\text{ Threats} \quad & \quad +qv\left(x\right)\phi \sum_{i=1}^{n(x)} \left[\boldsymbol{V}\left(t_U\left(x\right),i\right) - V\left(x,i\right)\right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE\text{ Hires} \quad & \quad +qv\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x\in\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - V\left(x,i\right)\right] dH_n\left(x',i'\right) \\
EE\text{ Threats} \quad & \quad +qv\left(x\right)\left(1-\phi\right) \sum_{i=1}^{n(x)} \int_{x\notin\mathcal{Q}^E(x',i')} \left[\boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - V\left(x,i\right)\right] dH_n\left(x',i'\right) \\
\text{Shocks} \quad & \quad + \sum_{i=1}^{n(x)} \Gamma_z[\boldsymbol{V},V](x,i)
\end{aligned}$$

Now sum, up all the previous terms, collect terms and use the definition of $\overline{V}(x)$:

$$
\begin{aligned}
\rho\overline{V}(x) &= \sum_{i=1}^{n(x)} w(x,i) \\
\text{Destructions} \quad & + \sum_{i=1}^{n(x)} \delta \left[U + \sum_{j\neq i}^{n(x)} \boldsymbol{V}(d(x,i),j) - \overline{V}(x) \right] \\
\text{Retentions} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - \overline{V}(x) \right] dH_v(x') \\
EE \text{ Quits} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - \overline{V}(x) \right] dH_v(x') \\
UE \text{ Hires} \quad & + qv(x)\,\phi \left[\sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_U(x),i\right) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE \text{ Threats} \quad & + qv(x)\,\phi \left[\sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_U(x),i\right) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE \text{ Hires} \quad & + qv(x)(1-\phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[\sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - \overline{V}(x) \right] dH_n(x',i') \\
EE \text{ Threats} \quad & + qv(x)(1-\phi) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - \overline{V}(x) \right] dH_n(x',i') \\
\text{Shocks} \quad & + \Gamma_z[\overline{\boldsymbol{V}},\overline{V}](x)
\end{aligned}
$$

Adding this last equation to the Bellman equation for $J(x)$ yields

$$
\begin{aligned}
\rho\Omega(x) &= y(x) - c(v(x),x) \\
\text{Destructions} \quad & + \sum_{i=1}^{n(x)} \delta \left[\boldsymbol{J}(d(x,i)) + U + \sum_{j\neq i}^{n(x)} \boldsymbol{V}(d(x,i),j) - J(x) - \overline{V}(x) \right] \\
\text{Retentions} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right),j\right) - J(x) - \overline{V}(x) \right] dH_v(x') \\
EE \text{ Quits} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{J}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(h_E\left(x,i,x'\right),i\right) + \sum_{j\neq i}^{n(x)} \boldsymbol{V}\left(q_E\left(x,i,x'\right),j\right) - J(x) - \overline{V}(x) \right] dH_v(x') \\
UE \text{ Hires} \quad & + qv(x)\,\phi \left[\boldsymbol{J}\left(h_U(x)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_U(x),i\right) - J(x) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\in\mathcal{A}\}} \\
UE \text{ Threats} \quad & + qv(x)\,\phi \left[\boldsymbol{J}\left(t_U(x)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_U(x),i\right) - J(x) - \overline{V}(x) \right] \cdot \mathbb{I}_{\{x\notin\mathcal{A}\}} \\
EE \text{ Hires} \quad & + qv(x)(1-\phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_E\left(x',i',x\right),i\right) - J(x) - \overline{V}(x) \right] dH_n(x',i') \\
EE \text{ Threats} \quad & + qv(x)(1-\phi) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{J}\left(t_E\left(x',i',x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(t_E\left(x',i',x\right),i\right) - J(x) - \overline{V}(x) \right] dH_n(x',i') \\
\text{Shocks} \quad & + \Gamma_z[\boldsymbol{J}+\overline{\boldsymbol{V}}, J+\overline{V}](x) - J(x) - \overline{V}(x)
\end{aligned}
$$

Collecting terms and using the definition of Ω :

$$\begin{aligned}
\rho\Omega(x) &= y(x) - c(v(x), x) \\
\text{Destructions} \quad & + \sum_{i=1}^{n(x)} \delta\left[\boldsymbol{\Omega}(d(x,i)) + U - \Omega(x)\right] \\
\text{Retentions} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega(x)\right] dH_v\left(x'\right) \\
EE \text{ Quits} \quad & + \lambda^E \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^E(x,i)} \left[\boldsymbol{\Omega}\left(q_E\left(x,i,x'\right)\right) + \boldsymbol{V}\left(h_E\left(x,i,x'\right), i\right) - \Omega(x)\right] dH_v\left(x'\right) \\
UE \text{ Hires} \quad & + qv(x)\,\phi\left[\boldsymbol{\Omega}(h_U(x)) - U - \Omega(x)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
UE \text{ Threats} \quad & + qv(x)\,\phi\left[\boldsymbol{\Omega}(t_U(x)) - \Omega(x)\right] \cdot \mathbb{I}_{\{x \notin \mathcal{A}\}} \\
EE \text{ Hires} \quad & + qv(x)(1-\phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \boldsymbol{V}\left(h_E\left(x',i',x\right), i'\right) - \Omega(x)\right] dH_n\left(x',i'\right) \\
EE \text{ Threats} \quad & + qv(x)(1-\phi) \int_{x \notin \mathcal{Q}^E(x',i')} \left[\boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right) - \Omega(x)\right] dH_n\left(x',i'\right) \\
\text{Shocks} \quad & + \Gamma_z[\overline{\boldsymbol{\Omega}}, \overline{\Omega}](x)
\end{aligned}$$

B.2 Value sharing

To make progress on (3), we begin by stating seven intermediate results, conditions **(C-RT)-(C-E)** which we prove from the assumptions listed in Section 2.2. These results establish how worker values $\boldsymbol{V}$ in (3) evolve in the six cases of hiring, retention, layoff, quits, exit and vacancy creation. Next, we apply conditions **(C-RT)-(C-E)** to (3).

To highlight the structure of the argument, we note a key implication our zero-sum game assumption **(A-IN)**: during internal negotiation, any value lost to one party must accrue to the other. This feature is obvious in the static model, and extends readily to our dynamic environment. In other words, the joint value of the firm plus its incumbent workers is invariant during the negotiation. We use this property extensively in the proof. This generalizes pairwise efficient bargaining—commonly used in one-worker firm models with linear production—to an environment with multi-worker firms and decreasing returns in production.

We now state the seven conditions that we apply to (3). In section B.3 below, we prove how each of them is implied by the assumptions of Section 2.2.

(C-RT) **Retentions and Threats.** First, if firm x meets an unemployed worker and the worker is not hired but only used as a threat, then the joint value of coalition x does not change since threats only redistribute value within the coalition. Second, when firm x uses employed worker i' from firm x' as a threat, the joint value of coalition x does not change. Third, when firm x meets worker i' at x' and the worker is retained by firm x', the joint value of coalition x' does not change. Formally,

$$\boldsymbol{\Omega}\left(r\left(x',i',x\right)\right) = \Omega(x') \quad , \quad \boldsymbol{\Omega}(t_U(x)) = \Omega(x) \quad , \quad \boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right) = \Omega(x).$$

Respectively, these imply that the *Retention*, *UE Threat* and *EE Threat* components of (3) are equal to zero.

(C-UE) *UE* **Hires.** An unemployed worker that meets firm x is hired when $x \in \mathcal{A}$. This set consists of firms that have a joint value after hiring that is higher than the pre-hire joint value plus the outside value of the hired worker. Due to the take-leave offer, the new hire receives her outside value, which is the value of unemployment:

$$\mathcal{A} = \{x | \boldsymbol{\Omega}(h_U(x)) - \Omega(x) \geq U\} \quad , \quad \boldsymbol{V}(h_U(x), i) = U.$$

(C-EE) EE **Hires.** An employed worker i' at firm x' that meets firm x is hired when $x \in \mathcal{Q}^E(x', i')$. This set consists of firms that have a higher marginal joint value than that of the current firm:

$$\mathcal{Q}^E(x', i') = \left\{ x \middle| \boldsymbol{\Omega}\left(h_E(x', i', x)\right) - \Omega(x) \geq \Omega(x') - \boldsymbol{\Omega}\left(q_E(x', i', x)\right) \right\}.$$

Due to the take-leave offer, the new hire receives her outside value, which is the marginal joint value at her current firm:

$$\boldsymbol{V}\left(h_E(x', i', x)\right) = \Omega(x') - \boldsymbol{\Omega}\left(q_E(x', i', x)\right).$$

(C-EU) EU **Quits and Layoffs.** An employed worker i at firm x quits to unemployment when $(x, i) \in \mathcal{Q}^U$. This set consist of states x such that the marginal joint value is less than the value of unemployment:

$$\begin{aligned} \mathcal{Q}^U &= \left\{ (x, i) \middle| \Omega\left(\hat{s}_{q1}(x, i)\right) + U > \Omega\left(\hat{s}_{q0}(x, i)\right) \right\}, \\ \text{where} \quad \hat{s}_{q1}(x, i) &= s\left(x, \left(1 - \left[q_{U,-i}(x); q_{U,i}(x) = 1\right]\right) \circ \left(1 - \ell(x)\right)\right), \\ \hat{s}_{q0}(x, i) &= s\left(x, \left(1 - \left[q_{U,-i}(x); q_{U,i}(x) = 0\right]\right) \circ \left(1 - \ell(x)\right)\right). \end{aligned}$$

The first expression captures when worker i quits, and the second where worker i does not. Similarly, an EU layoff will be chosen by the firm when $(x, i) \in \mathcal{L}$:

$$\begin{aligned} \mathcal{L} &= \left\{ (x, i) \middle| \Omega\left(\hat{s}_{\ell 1}(x, i)\right) + U > \Omega\left(\hat{s}_{\ell 0}(x, i)\right) \right\}, \\ \text{where} \quad \hat{s}_{\ell 1}(x, i) &= s\left(x, \left(1 - \left[\ell(x); \ell_i(x) = 1\right]\right) \circ \left(1 - q_U(x)\right)\right), \\ \hat{s}_{\ell 0}(x, i) &= s\left(x, \left(1 - \left[\ell(x); \ell_i(x) = 0\right]\right) \circ \left(1 - q_U(x)\right)\right). \end{aligned}$$

The first expression captures when worker i is laid off, and the second when worker i is not.

(C-X) **Exit.** A firm x exits when $x \in \mathcal{E}$. This set consists of the states in which the total outside value of the firm and its workers is larger than the joint value of operation:

$$\mathcal{E} = \left\{ x \middle| \vartheta + n\left(s\left(x, \kappa(x)\right)\right) \cdot U > \Omega\left(s\left(x, \kappa(x)\right)\right) \right\}.$$

(C-V) **Vacancies.** The expected return to a matched vacancy $R(x)$ depends only on the joint value, and so the firm's optimal vacancy policy $v(x)$ depends only on the joint value. The policy $v(x)$ solves

$$\max_v \; q(\theta) v R(x) - c(v, x),$$

where the expected return to a matched vacancy is

$$\begin{aligned} R(x) &= \phi \underbrace{\left[\boldsymbol{\Omega}\left(h_U(x)\right) - \Omega(x) - U\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}}_{\text{Return from unemployed worker match}} \\ &+ (1 - \phi) \underbrace{\int_{x \in \mathcal{Q}^E(x', i')} \left\{ \left[\boldsymbol{\Omega}\left(h_E(x', i', x)\right) - \Omega(x)\right] - \left[\Omega(x') - \boldsymbol{\Omega}\left(q_E(x', i', x)\right)\right] \right\} dH_n(x', i')}_{\text{Expected return from employed worker match}}. \end{aligned}$$

(C-E) **Entry.** A firm enters if and only if

$$\int \boldsymbol{\Omega}(x_0)\, d\Pi_0(z) \geq c_0 + n_0 U.$$

Summarizing (C). The substantive result is that all firm and worker decisions and employed workers' values can be expressed in terms of joint value Ω and exogenous worker outside option U.

B.3 Proof of Conditions (C)

B.3.1 Proof of C-UE and C-RT (UE Hires and UE Threats)

In this subsection, we consider a meeting between a firm x and an unemployed worker. Following **A-IN** and **A-EN**, the firm internally renegotiates according to a zero-sum game with its incumbent workers and makes a take-leave offer to the new worker. Intuitively, having the worker "at the door" is identical to having her hired at value U for the firm and for all incumbent workers: the firm can always make new take-leave offers to its incumbents after hiring the new worker. Hence, we expect the firm to make one take-leave offer to the new worker and its incumbents at the time of the meeting, and not make a new, different offer to is incumbents afer hiring has taken place.

We start by showing this equivalence formally. To do so, when meeting an unemployed worker, we let the firm conduct internal renegotiation with its incumbent workers and make an offer to the new worker. Then, we let a second round of internal offers take place after the hiring. We introduce some notation to keep track of values throughout the internal and external negotiations. To fix ideas, we denote by (IR1) the first round of internal negotiation, pre-external negotiation. We denote by (IR2) the second round of internal negotiation, post-hire.

Post-hire and post-internal negotiation (IR2) values are denoted with double stars. Post-internal-negotiation (IR1) but pre-external-negotiation values are denoted with stars.

$$\Omega^{**} := J^{**} + \sum_{j=1}^{n(x)} V_j^{**} + V_i^{**} \qquad \Omega^{*} := J^{*} + \sum_{j=1}^{n(x)} V_j^{*} \qquad \Omega := J + \sum_{j=1}^{n(x)} V_j$$

Proceeding by backward induction, under **A-EN** the firm makes a take-it-or-leave-it offer to the unemployed worker, therefore

$$V_i^{**} = U$$

We now divide the proof in several steps. We start by proving that for all incumbent workers $j = 1...n(x)$, $V_j^{**} = V_j^{*}$. We then use **A-IN** to argue that $\Omega^* = \Omega$. Once these claims have been proven, we move on to proving **C-UE** (UE Hires) and the part of for threats from unemployment **C-RT** (UE Threats). Finally, we show that our microfoundations for the renegotiation game deliver **A-IN**.

Claim 1: For all incumbents workers $j = 1...n(x)$, we have $V_j^{**} = V_j^{*}$.

We proceed by backwards induction using our assumptions **A-EN** and **A-IN**. Immediately after (IR1) has taken place, only the following events can happen:

1. Hire/not-hire
   - Either the worker is hired from unemployment (H),
   - Or the worker is not hired from unemployment (NH)
2. Possible new round of internal negotiation (IR2). This possible second round of internal negotiation (now including the newly hired worker) leads to values V_j^{**}.

We focus on subgame perfect equilibria in this multi-stage game. Therefore, after (IR1), workers perfectly anticipate what the outcome of the hire/not-hire stage will be. That is, after (IR1), they know perfectly what hiring decision (H or NH) the firm will make. Now suppose that internal renegotiation (IR2) actually happens after the hire/not-hire decision, that is, that for some incumbent worker $j \in \{1, ..., n(x)\}$, $V_j^{**} \neq V_j^{*}$. The firm has no incentives to accept a change in the new worker's value to anything above U, so by **A-MC** her value does not change in the second round (IR2).

We construct the rest of the proof by contradiction. Consider for a contradiction an incumbent worker j whose value changed in (IR2). Because of **A-MC**, her value can change only in the following cases:

- The firm has a credible threat to fire worker j, in which case $V_j^{**} < V_j^*$
- Worker j has a credible threat to quit, in which case $V_j^{**} > V_j^*$

In addition, those credible threats can lead to a different outcome than in (IR1), and thus $V_j^{**} \neq V_j^*$, only if the threat on either side was not available in (IR1). If that same threat was available in the first round (IR1), then the outcome of the bargaining (IR1) would have been V_j^{**}.

Recall that both incumbent worker j and the firm understand and anticipate which hire/not-hire decision the firm will make after the first round (IR1). They also understand and anticipate that, in case of hire, the value of the new worker will remain U in the second round (IR2).

Therefore, the firm can *credibly threaten* to hire the new worker in the first round *if and only if* it actually hires her after the first round (IR1) is over. This implies that the firm can credibly threaten worker to fire j in the second round (IR2), by **A-LC**, *if and only if* it could credibly threaten her with hiring the new worker *in the first round of internal renegotiation (IR1).* This in turn entails that any credible threat the firm can make in the second round (IR2) was already available in the first round.

On the worker side, quitting into unemployment is a credible threat when her value is below the value of unemployment. So this threat does not change between the first round (IR1) and the second round (IR2), because the equilibrium value to that worker will always be above the value of unemployment.

In sum, the set of credible threats both to the firm and to worker j does not change between the initial round of internal renegotiation (IR1) and the post-hiring-decision round (IR2). This finally implies that the outcome of the initial round of internal renegotiation (IR1) for any incumbent j remains unchanged in the second round (IR2), that is:

$$V_j^{**} = V_j^*$$

which proves **Claim 1**.

We can now move on to proving **C-UE**.

Proof of C-UE. Using the definitions of Ω^{**} and Ω, we can write

$$\Omega^{**} - \Omega = \left[J^{**} + \sum_{j=1}^{n(x)} V_j^{**} + V_i^{**} \right] - \left[J + \sum_{j=1}^{n(x)} V_j \right]$$

Now using $V_i^{**} = U$, we obtain

$$\Omega^{**} - \Omega = \left[J^{**} + \sum_{j=1}^{n(x)} V_j^{**} \right] - \left[J + \sum_{j=1}^{n(x)} V_j \right] + U$$

Using **Claim 1:** $V_j^{**} = V_j^*$, and adding and subtracting J^* we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] + \left[J^* + \sum_{j=1}^{n(x)} V_j^* \right] - \left[J + \sum_{j=1}^{n(x)} V_j \right] + U$$

Subsituting in the definition of Ω and of Ω^*,

$$\Omega^{**} - \Omega = [J^{**} - J^*] + [\Omega^* - \Omega] + U$$

Finally recall that internal renegotiation is (1) individually rational, and (2) is a zero-sum game, according to **A-IN**. Thus, all incumbent workers remain in the coalition after internal renegotiation, and the joint value is unchanged:

$\Omega^* = \Omega$. Using $\Omega^* = \Omega$

$$\Omega^{**} - \Omega = [J^{**} - J^*] + U$$

which can be re-written

$$J^{**} - J^* = [\Omega^{**} - \Omega] - U$$

Now under **A-LC**, the firm will only hire if its value after hiring is higher than its value after internal renegotiation: $J^{**} - J^* \geq 0$. This inequality requires

$$\Omega^{**} - \Omega \geq U$$
$$\mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) \geq U$$

The firm does not hire when its value of hiring is below its value of renegotiation $J^{**} < J^*$. This inequality implies

$$\Omega^{**} - \Omega < U$$

When the firm does not hire, we obtain using again **A-IN** and $\Omega^* = \Omega$:

$$\Omega^{**} - \Omega^* < U$$

which finally implies

$$\mathbf{\Omega}\left(h_U\left(x\right)\right) - \mathbf{\Omega}\left(t_U(x)\right) < U$$

Now, we argue that conditional on not hiring, $\Omega^{**} = \Omega^* = \Omega$, where in this case Ω^{**} denotes the value of the coalition without hiring, and thus does not include the value of the unemployed worker. Just as before, this is a direct consequence from **A-IN** and that the internal renegotiation game is zero-sum.

Therefore:

$$\mathbf{\Omega}\left(t_U(x)\right) = \Omega(x)$$

We have therefore shown **C-UE** and part of **C-RT (**UE **Hires and** UE **Threats)**: An unemployed worker that meets x is hired when$x \in \mathcal{Q}^U$, where

$$\mathcal{A} = \left\{x \middle| \mathbf{\Omega}\left(h_U\left(x\right)\right) - \Omega\left(x\right) \geq U\right\}$$

and upon joining the firm, has value

$$\boldsymbol{V}\left(h_U\left(x, i\right)\right) = U.$$

and

$$\mathbf{\Omega}(t_U(x)) = \Omega(x).$$

B.3.2 Proof of C-EE and C-RT (EE Hires, EE Threats and Retentions)

The structure of the argument for EE hires, threats and retentions follows closely the steps of the argument for UE hires, threats and retentions. The only major difference is that the worker's outside option is endogenously determined when hired from employment. Consider firm x that has met worker i' at firm x'. We proceed in two steps.

Maximum value at incumbent firm. We first seek to determine the maximum value that x' may offer to its worker when it may be poached by firm x. Under **A-IN** and **A-EN**, upon meeting an employed worker, internal negotiation may take place at the poaching firm x, and x makes a take-it-or-leave-it offer. Internal negotiation may take place at x' with all workers including i'.

Proceeding by backward induction, we again define intermediate values but here at x', noting that $q_E\left(x', i', x\right)$ gives the number of employees in x' if the worker leaves:

$$\Omega = J + \sum_{j=1}^{n(q_E(x',i',x))} V_j + V_{i'} \qquad \Omega^* = J^* + \sum_{j=1}^{n(q_E(x',i',x))} V_j^* + V_{i'}^* \qquad \Omega^{**} = J^{**} + \sum_{j=1}^{n(q_E(x',i',x))} V_j^{**}$$

In the second equation we are describing the values of the firm in renegotiation where i' stays with the firm, so $V_{i'}^*$ is the outcome of internal negotiation. In the third equation we consider the firm having lost the worker. Under **A-EN** the firm will respond to an offer $\overline{V}$ from x with

$$V_{i'}^* = \overline{V}$$

Following the same arguments as in **Claim 1** from section B.3.1, the same result obtains: under **A-EN** and **A-IN**, the values accepted by the incumbent workers *after the internal renegotiation* $(V_j^*)_j$ will be equal to the values they receive *after the external negotiation* $(V_j^{**})_j$, that is

$$V_j^{**} = V_j^*$$

Then following the same steps as in section B.3.1, and using again that $\Omega^* = \Omega$, we obtain that:

$$\Omega^{**} - \Omega = [J^{**} - J^*] - \overline{V}$$

Now under **A-LC**, the firm x' will only try to keep the worker if $J^* > J^{**}$, which requires

$$\Omega - \Omega^{**} \leq \overline{V} \qquad\qquad \boldsymbol{\Omega}\left(r(x',i',x) - \boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right) \leq \overline{V}\right.$$

This determines the maximum value that x' can offer to the worker to retain them.

Poaching. Our second step is to check when worker i' moves from x' to x. The bargaining protocol implies that x firm will offer $\overline{V}$ if it is making an offer, since it need not offer more. For firm x the argument may proceed identically to the case of unemployment, simply replacing U with $\overline{V}$. The result is that the firm will hire only if and only if

$$\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right) \geq \overline{V}$$

or, equivalently,

$$\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right) \geq \boldsymbol{\Omega}\left(r(x',i',x)\right) - \boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)$$

When firm x does not hire, **A-IN** applies, and so

$$\boldsymbol{\Omega}(t_E(x',i',x)) = \Omega(x)$$

Similarly, when firm x' is not poached, **A-IN** applies, and so

$$\boldsymbol{\Omega}(r(x',i',x)) = \Omega(x')$$

The combination of these conditions deliver **C-UE** and part of **C-RT** (*EE* Hires, *EE* Threats and Retention):

1. The quit set of an employed worker is determined by

$$\mathcal{Q}^E\left(x',i'\right) = \left\{ x \middle| \boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right) \geq \Omega\left(x'\right) - \boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right) \right\}$$

2. The worker's value of being hired from employment from firm x' is

$$\boldsymbol{V}(h_E(x,x',i')) = \Omega\left(x'\right) - \boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)$$

3. Worker i's value of being retained at x' after meeting x is[3]

$$\boldsymbol{V}(r(x',i',x),i') = \boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right) - \Omega\left(x\right)$$

4. The joint value of the potential poaching firm x when the worker is not hired does not change:

$$\boldsymbol{\Omega}(t_E(x',i',x)) = \Omega(x)$$

5. The joint value of the potential poached firm x' does not change when the worker stays:

$$\boldsymbol{\Omega}(r(x',i',x)) = \Omega(x')$$

B.3.3 Proof of C-EU (EU Quits and layoffs)

We first derive our expression for $\mathcal{L}$ on the firm side, and next our expression for $\mathcal{Q}^U$ on the worker side.

Part 1: Firm side Consider a firm x who is considering laying off worker i for whom $q_{U,i}(x) = 0$. As above, we start with definitions, noting that $n(s(\cdot)) \equiv n\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=1\right]\right)\circ\left(1-q_U\left(x\right)\right)\right)\right)$ is the number of workers if i is laid off:

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i \qquad \Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^* \qquad \Omega^{**} = J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}$$

Note that in the first line the coalition has still worker i in it. In the second line, the firm and the worker i have negotiated (and internal negotiation has determined V_i^* which is what i will get if they stay in the firm). In the third line, the worker has been fired and another round of negotiation has occurred among incumbents.

The same result as in **Claim 1** from section B.3.1 obtains: under **A-BP**, $V_j^{**} = V_j^*$. Using this result and the above definitions as before,

$$\Omega^{**} - \Omega = [J^{**} - J^*] + [\Omega^* - \Omega] - V_i^*$$

Using again **A-IN** to conclude that $\Omega^* = \Omega$, we obtain

$$\Omega^{**} - \Omega = [J^{**} - J^*] - V_i^*$$

Now under **A-LC**, the firm x will only layoff the worker if $J^{**} > J^*$, which requires

$$\Omega - \Omega^{**} < V_i^*$$

As long as $V_i^* > U$ the worker would be willing to renegotiate and transfer value to the firm to avoid being laid off, implying

$$\Omega - \Omega^{**} < U.$$

which we can re-write

$$\Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=1\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),i\right) + U > \Omega\left(s\left(x,\left(1-\left[\ell\left(x\right);\ell_i\left(x\right)=0\right]\right)\circ\left(1-q_U\left(x\right)\right)\right),i\right)$$

where the LHS is $\Omega^{**} + U$ (under the layoff) and the RHS is Ω. This concludes the proof for the firm side.

[3] Because offers are made at no cost, both firms always make an offer, even when they know that they cannot retain/hire the worker in equilibrium. This is exactly the same as in Postel-Vinay Robin (2002).

Part 2: Worker side Consider worker i in firm x who is considering quitting to unemployment for whom $\ell_i(x) = 0$. As above, we start with definitions, noting that $n(s(\cdot)) \equiv n(s(x, (1-\ell(x)) \circ (1 - [q_{U,-i}(x); q_{U,i}(x) = 1])))$ is the number of workers if i quits. As before, we define

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j + V_i \qquad \Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* + V_i^* \qquad \Omega^{**} = J^{**} + \sum_{j=1}^{n(s(\cdot))} V_j^{**}$$

The same result as in **Claim 1** from section B.3.1 obtains $V_j^{**} = V_j^*$. Using this result and the above definitions,

$$\Omega^{**} - \Omega = [J^{**} - J^*] + [\Omega^* - \Omega] - V_i^*$$

Again, $\Omega^* = \Omega$ from **A-IN**, so that

$$\Omega^{**} - \Omega = [J^{**} - J^*] - V_i^*$$

Now under **A-LC**, worker i will quit into unemployment iff $V_i^* < U$, which requires

$$J^{**} - J^* + [\Omega - \Omega^{**}] < U$$

As long as $J^{**} < J^*$, the firm is willing to transfer value to worker i to retain her. Therefore, worker i quits into unemployment iff the previous inequality holds at $J^{**} = J^*$, i.e.

$$\Omega - \Omega^{**} < U$$

Therefore, the worker quits iff

$$\begin{aligned} &\Omega(s(x, (1-\ell(x)) \circ (1 - [q_{U,-i}(x); q_{U,i}(x) = 1])), i) + U \\ &> \Omega(s(x, (1-\ell(x)) \circ (1 - [q_{U,-i}(x); q_{U,i}(x) = 0])), i) \end{aligned}$$

which concludes the proof of the worker side. This delivers **C-EU**.

B.3.4 Proof of C-X (Exit)

Consider a firm x who contemplates exit after all endogenous quits and layoffs, thus when its employment is $n(s(x, \kappa(x)))$. As before we define values conditional on exiting:

$$\Omega = J + \sum_{j=1}^{n(s(\cdot))} V_j \qquad \Omega^* = J^* + \sum_{j=1}^{n(s(\cdot))} V_j^* \qquad \Omega^{**} = J^{**} + 0$$

The joint value after exit is simply the value of the firm, since all other workers have left because of exit. Following similar calculations as before,

$$\Omega^{**} - \Omega \quad = \quad [J^{**} - J^*] + [\Omega^* - \Omega] - \sum_{j=1}^{n(s(\cdot))} V_j^*$$

Again, $\Omega^* = \Omega$ from **A-IN**, so that

$$\Omega^{**} - \Omega = [J^{**} - J^*] - \sum_{j=1}^{n(s(\cdot))} V_j^*$$

The firm exits iff $J^{**} \geq J^{*}$, that is, $\vartheta \geq J^{*}$. This is equivalent to

$$\Omega^{**} - \Omega \geq - \sum_{j=1}^{n(s(\cdot))} V_j^{*}$$

Using again that $\Omega^{**} = J^{**} = \vartheta$, the firm exits iff

$$\vartheta + \sum_{j=1}^{n(s(\cdot))} V_j^{*} \geq \Omega$$

Since any worker is better off under $V_i^{*} \geq U$ than unemployed, all workers are willing to take a value cut down to U if $\vartheta \geq \Omega - \sum_{j=1}^{n(s(\cdot))} V_j^{*}$ because then the firm can credibly exit. This observation implies that the firm exits if and only if

$$\vartheta - \Omega\left(s\left(x, \kappa\left(x\right)\right)\right) + n\left(s\left(x, \kappa\left(x\right)\right)\right) U \geq 0$$

This last equality proves **C-X (Exit)**: the set of x such that the firm exits is given by

$$\mathcal{E} = \left\{ x \middle| \vartheta + n\left(s\left(x, \kappa\left(x\right)\right)\right) \cdot U \geq \Omega\left(s\left(x, \kappa\left(x\right)\right)\right) \right\}$$

B.3.5 Proof of C-V (Vacancies)

We split the proof in two steps. First, we show that workers are collectively willing to transfer value to the firm in exchange for the joint value-maximizing vacancy policy function. Second, we show that a single worker can create a system of transfers that achieves the same outcome. These transfers are equivalent to wage renegotiation, which explains why we have subsumed them in the wage function $w(x, i)$ in the equations above. Similarly to wages, these transfers drop out from the expression for the joint value.

Part 1: Collective transfers In this step, we show that workers are collectively better off transferring value to the firm in exchange of the firm posting the joint value-maximizing amount of vacancies.

The vacancy posting decision v^{J} that maximizes firm value is:

$$\frac{c_v\left(v^{J}\left(x\right), n\left(x\right)\right)}{q} = \phi\left[\boldsymbol{J}\left(h_U\left(x\right)\right) - J\left(x\right)\right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} + (1-\phi) \int_{x \in \mathcal{Q}^{E}(x', i')} \left[\boldsymbol{J}\left(h_E\left(x', i', x\right)\right) - J\left(x\right)\right] dH_n\left(x', i'\right).$$

Similarly, define v^{Ω} be the policy that maximizes the value of the coalition, and $v^{\overline{V}}$ be the policy that maximizes the value of all the employees. Let $\Omega^{\gamma}, J^{\gamma}, \overline{V}^{\gamma}$ be the value of the coalition, firm and all workers under the v^{γ}, for $\gamma \in \left\{\Omega, J, \overline{V}\right\}$. We now prove our claim in several steps.

Part 1-(a) Collective value gains. The policy v^{Ω} will lead to $\overline{V}^{\Omega} \geq \overline{V}^{J} + [J^{J} - J^{\Omega}]$ where $J^{J} - J^{\Omega} \geq 0$.

Proof: By construction Ω^{Ω} is greater than Ω^{J}: $\Omega^{\Omega} \geq \Omega^{J}$. By definition: $\Omega^{\Omega} = J^{\Omega} + \overline{V}^{\Omega}$, and $\Omega^{J} = J^{J} + \overline{V}^{J}$. Use those definitions to obtain inequality $J^{\Omega} + \overline{V}^{\Omega} \geq J^{J} + \overline{V}^{J}$, which can be re-arranged into $\overline{V}^{\Omega} - \overline{V}^{J} \geq J^{J} - J^{\Omega}$. Since J^{J} is the value under the optimal policy for J, then $J^{J} \geq J^{\Omega}$. The above then implies that

$$\overline{V}^{\Omega} - \overline{V}^{J} \geq J^{J} - J^{\Omega} \geq 0$$

This inequality implies that workers would be prepared to transfer $T = J^{J} - J^{\Omega} \geq 0$ to the firm in order for the firm to pursue policy v^{Ω} instead of v^{J}. This concludes the proof of **Part 1-(a).**

Part 1-(b) Infeasibility of $\overline{V}^{\overline{V}}$. There does not exist an incentive-compatible transfer from workers to firm that will lead to $\overline{V}^{\overline{V}}$.

Proof: Suppose workers consider transferring even more to induce the firm to follow policy $v^{\overline{V}}$ that maximizes their value. By construction $\Omega^{\Omega} \geq \Omega^{\overline{V}}$. Using definitions for each of these, then $J^{\Omega} + \overline{V}^{\Omega} \geq J^{\overline{V}} + \overline{V}^{\overline{V}}$. Rearranging this: $J^{\Omega} - J^{\overline{V}} \geq \overline{V}^{\overline{V}} - \overline{V}^{\Omega}$. Since $\overline{V}^{\overline{V}}$ is the value under the optimal policy for $\overline{V}$, then $\overline{V}^{\overline{V}} \geq \overline{V}^{\Omega}$. The above then implies that

$$J^{\Omega} - J^{\overline{V}} \geq \overline{V}^{\overline{V}} - \overline{V}^{\Omega} \geq 0$$

Taking v^{Ω} as a baseline, the above implies that a change to $v^{\overline{V}}$ causes a loss of $J^{\Omega} - J^{\overline{V}}$ to the firm, which is more than the gain of $\overline{V}^{\overline{V}} - \overline{V}^{\Omega}$ to the workers. This implies that workers could transfer all of their gains under $v^{\overline{V}}$ to the firm, but the firm would still not choose $v^{\overline{V}}$ over v^{Ω}. This concludes the proof of **Part 1-(b).**

Part 1-(c) Optimality of $\overline{V}^{\Omega}$. There does not exist an incentive-compatible transfer from workers to firm that will lead to $\overline{V}^{*} \in \left(\overline{V}^{\Omega}, \overline{V}^{\overline{V}}\right)$.

Proof: Call such a policy $v^{\overline{V}*}$. Then: $\Omega^{\Omega} \geq \Omega^{\overline{V}*}$, and by definition,

$$J^{\Omega} - J^{\overline{V}^*} \quad \geq \quad \overline{V}^{\overline{V}^*} - \overline{V}^{\Omega}$$

Since by definition $\overline{V}^{*} \in \left(\overline{V}^{\Omega}, \overline{V}^{\overline{V}}\right)$, then $\overline{V}^{\overline{V}^*} - \overline{V}^{\Omega} \geq 0$. Therefore

$$J^{\Omega} - J^{\overline{V}^*} \geq \overline{V}^{\overline{V}^*} - \overline{V}^{\Omega} \geq 0$$

Taking v^{Ω} as a baseline, the above implies that a change to $v^{\overline{V}^*}$ causes a loss of $J^{\Omega} - J^{\overline{V}^*}$ to the firm, which is more than the gain of $\overline{V}^{\overline{V}^*} - \overline{V}^{\Omega}$ to the workers. This concludes the proof of **Part 1-(c).**

Part 1-(d) Conclusion. In summary, it is optimal for workers to transfer exactly $T = J^J - J^{\Omega}$ to the firm, in order for the firm to pursue v^{Ω} instead of v^J. Further transfers to the firm would be required to have the firm pursue a better policy for workers, but this is exceedingly costly to the firm and the workers are unwilling to make a transfer to cover these costs. This concludes the proof of **Step 1: Collective transfers.**

Part 2: Individual transfers In this step, we show that a single, randomly drawn worker can construct a system of transfers that induces the firm to post v^{Ω} instead of v^J, while leaving all agents better off.

Within dt, consider the single, randomly drawn worker j_0. Consider the following system of transfers. Worker j_0 makes a transfer $J^J - J^{\Omega}$ to the firm, in exchange of what (i) the firm posts v^{Ω} instead of v^J, and (ii) the worker gets a wage increase that gives her all the differential surplus $\overline{V}^{\Omega} - \overline{V}^{J}$.

Following the same steps as in **Part 1: Collective transfers**, the firm gets $J^{\Omega} + [J^J - J^{\Omega}] = J^J$ and is hence indifferent. Similarly, workers $j \neq j_0$ do not get any value change, and are thus indifferent Finally, worker j_0 gets a value increase of

$$[\overline{V}^{\Omega} - \overline{V}^{J}] - [J^J - J^{\Omega}] \geq 0$$

where the inequality similarly follows from **Part 1: Collective transfers**. This concludes the proof of **Part 2: Individual transfers**.

Conclusion. The previous arguments show that a single worker has an incentive to and can induce the firm to post v^{Ω}. Notice also that the same argument holds starting from any vacancy policy function $\widetilde{v} \neq v^J$ together with a value of the firm $\widetilde{J}$. Thus, even if some worker induces the firm to post a different vacancy policy function which is not v^{Ω} any other worker has an incentive to induce the firm to post v^{Ω}. Therefore, in equilibrium, the firm posts v^{Ω}, which concludes the proof of **C-V**.

B.4 Applying Conditions (C)

Having established that **Assumption (A)** can be used to prove **Conditions (C)**, we now apply conditions **(C)** to the Bellman equation for the joint value. The goal of this section is to show that for $x \in \mathcal{E}^c$ the complement of the exit set, we can considerably simplify the recursion for the joint value:

$$\begin{aligned}
\rho\Omega(x) &= y(z(x), n(x)) - c(v(x), n(x), z(x)) \\
\text{Destructions} \quad & -\delta \sum_{i=1}^{n(x)} [\Omega(x) - \mathbf{\Omega}(d(x,i)) - U] \\
UE \text{ Hires} \quad & +qv(x)\phi [\mathbf{\Omega}(h_U(x)) - \Omega(x) - U] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
EE \text{ Hires} \quad & +qv(x)(1-\phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[\left[\mathbf{\Omega}\left(h_E(x',i',x)\right) - \Omega(x) \right] - \left[\Omega(x') - \mathbf{\Omega}\left(q_E(x',i',x)\right) \right] \right] dH_n(x',i') \\
\text{Shocks} \quad & +\Gamma[\mathbf{\Omega}, \Omega] \qquad (4)
\end{aligned}$$

with the sets

$$\begin{aligned}
\mathcal{Q}^U &= \Big\{ (x,i) \Big| \Omega\left(s\left(x, (1-\ell(x)) \circ (1 - [q_{U,-i}(x); q_{U,i}(x) = 1])\right), i\right) + U \\
&\qquad > \Omega\left(s\left(x, (1-\ell(x)) \circ (1 - [q_{U,-i}(x); q_{U,i}(x) = 0])\right), i\right) \Big\} \qquad (5) \\
\mathcal{L} &= \Big\{ (x,i) \Big| \Omega\left(s\left(x, (1 - [\ell(x); \ell_i(x) = 1]) \circ (1 - q_U(x))\right), i\right) + U \\
&\qquad > \Omega\left(s\left(x, (1 - [\ell(x); \ell_i(x) = 0]) \circ (1 - q_U(x))\right), i\right) \Big\} \qquad (6) \\
\mathcal{E} &= \Big\{ x \Big| \vartheta + n(s(x, \kappa(x))) \cdot U \geq \Omega(s(x, \kappa(x))) \Big\} \qquad (7) \\
\mathcal{A} &= \Big\{ x \Big| \mathbf{\Omega}(h_U(x)) - \Omega(x) \geq U \Big\} \qquad (8) \\
\mathcal{Q}^E(x',i') &= \left\{ x \middle| \mathbf{\Omega}\left(h_E(x',i',x)\right) - \Omega(x) \geq \Omega(x') - \mathbf{\Omega}\left(q_E(x',i',x)\right) \right\} \qquad (9)
\end{aligned}$$

and—as per **(C-V)—**the vacancy policy $v(x)$ is given by the solution to the following:

$$\begin{aligned}
\frac{c_v(v(x), n(x))}{q} &= \phi [\mathbf{\Omega}(h_U(x)) - \Omega(x)] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}} \\
&+ (1-\phi) \int_{x \in \mathcal{Q}^E(x',i')} \left[\left[\mathbf{\Omega}\left(h_E(x',i',x)\right) - \Omega(x) \right] - \left[\Omega(x') - \mathbf{\Omega}\left(q_E(x',i',x)\right) \right] \right] dH_n(x',i') \qquad (10)
\end{aligned}$$

In continuous time, the exit decision is captured by $x \in \mathcal{E}$. The Bellman equation above holds exactly for $x \in \mathcal{E}^c$. Exit is accounted for in the "bold" continuation values, which all include the possible exit decision should the firm's state fall into $\mathcal{E}$ after an event.

We first proceed one term at the time, working through (B.4.1) exogenous destructions, (B.4.2) retentions, (B.4.3) *EE* (poached) quits, (B.4.4) *UE* hires, (B.4.5) *UE* threats, (B.4.6) *EE* (poached) hires, and (B.4.7) *EE* threats.

B.4.1 Exogenous destructions

$$\text{Destructions} = \sum_{i=1}^{n(x)} \delta \left[\boldsymbol{J}\left(d(x,i)\right) + \sum_{j=1}^{n(d(x,i))} \boldsymbol{V}\left(d(x,i), j\right) + U - \Omega\left(x\right) \right] = \sum_{i=1}^{n(x)} \delta \left[\boldsymbol{\Omega}\left(d(x,i)\right) + U - \Omega\left(x\right) \right]$$

where we simply have used the definition $\boldsymbol{\Omega}\left(d(x,i)\right) := \boldsymbol{J}\left(d(x,i)\right) + \sum_{j=1}^{n(d(x,i))} \boldsymbol{V}\left(d(x,i), j\right)$.

B.4.2 Retentions

$$\begin{aligned}
\text{Retentions} &= \lambda^{E} \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^{E}(x,i)} \left[\boldsymbol{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right), j\right) - \Omega\left(x\right) \right] dH_{v}\left(x'\right) \\
&= \lambda^{E} \sum_{i=1}^{n(x)} \int_{x' \notin \mathcal{Q}^{E}(x,i)} \left[\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) - \Omega\left(x\right) \right] dH_{v}\left(x'\right)
\end{aligned}$$

where we simply have used the definition $\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) = \boldsymbol{J}\left(r\left(x,i,x'\right)\right) + \sum_{j=i}^{n(x)} \boldsymbol{V}\left(r\left(x,i,x'\right), j\right)$. Now using the result in **C-RT** that

$$\boldsymbol{\Omega}\left(r\left(x,i,x'\right)\right) = \Omega(x')$$

we obtain that

$$\text{Retentions} = 0$$

B.4.3 *EE* Quits

$$EE \text{ Quits} = \lambda^{E} \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^{E}(x,i)} \left[\boldsymbol{J}\left(q_{E}\left(x,i,x'\right)\right) + \boldsymbol{V}\left(q_{E}\left(x,i,x'\right), i\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_{E}\left(x,i,x'\right), j\right) - \Omega\left(x\right) \right] dH_{v}\left(x'\right)$$

Now by definition

$$\boldsymbol{\Omega}\left(q_{E}\left(x,i,x'\right)\right) = \boldsymbol{J}\left(q_{E}\left(x,i,x'\right)\right) + \sum_{j=1}^{n(q_{E}(x,i,x'))} \boldsymbol{V}\left(q_{E}\left(x,i,x'\right), j\right) = \boldsymbol{J}\left(q_{E}\left(x,i,x'\right)\right) + \sum_{j \neq i}^{n(x)} \boldsymbol{V}\left(q_{E}\left(x,i,x'\right), j\right)$$

Using this last equality in the term in square brackets

$$EE \text{ Quits} = \lambda^{E} \sum_{i=1}^{n(x)} \int_{x' \in \mathcal{Q}^{E}(x,i)} \left[\Omega\left(q_{E}\left(x,i,x'\right)\right) - \Omega\left(x\right) + \boldsymbol{V}\left(q_{E}\left(x,i,x'\right), i\right) \right] dH_{v}\left(x'\right)$$

Using **C-EE**, the value going to the poached worker is $\boldsymbol{V}\left(q_{E}\left(x,i,x'\right)\right) = \Omega\left(x\right) - \Omega\left(q_{E}\left(x,i,x'\right)\right)$. Substituting this into the last equation, we observe that the term in the square brackets is zero, and so

$$EE \text{ Quits} = 0$$

B.4.4 *UE* Hires

$$UE \text{ Hires} = qv\left(x\right)\phi \left[\boldsymbol{J}\left(h_{U}\left(x\right)\right) + \sum_{i=1}^{n(x)} \boldsymbol{V}\left(h_{U}\left(x\right), i\right) - \Omega\left(x\right) \right] \cdot \mathbb{I}_{\{x \in \mathcal{A}\}}$$

Now, by definition

$$\boldsymbol{\Omega}\left(h_U\left(x\right)\right)=\boldsymbol{J}\left(h_U\left(x\right)\right)+\sum_{i=1}^{n(h_U(x))}\boldsymbol{V}\left(h_U\left(x\right),i\right)=\boldsymbol{J}\left(h_U\left(x\right)\right)+\sum_{i=1}^{n(x)}\boldsymbol{V}\left(h_U\left(x\right),i\right)+\boldsymbol{V}\left(h_U\left(x\right),i\right)$$

and so, re-arranging,

$$\boldsymbol{J}\left(h_U\left(x\right)\right)+\sum_{i=1}^{n(x)}\boldsymbol{V}\left(h_U\left(x\right),i\right)=\boldsymbol{\Omega}\left(h_U\left(x\right)\right)-\boldsymbol{V}\left(h_U\left(x\right),i\right)$$

Substituting this last equation into the term in the square brackets of the first equation,

$$UE\text{ Hires}=qv\left(x\right)\phi\left[\boldsymbol{\Omega}\left(h_U\left(x\right)\right)-\Omega\left(x\right)-\boldsymbol{V}\left(h_U\left(x\right),i\right)\right]\cdot\mathbb{I}_{\{x\in\mathcal{A}\}}$$

Following **C-UE**, the value going to the hired worker is $\boldsymbol{V}\left(h_U\left(x\right),i\right)=U$. Substituting in:

$$UE\text{ Hires}=qv\left(x\right)\phi\left[\boldsymbol{\Omega}\left(h_U\left(x\right)\right)-\Omega\left(x\right)-U\right]\cdot\mathbb{I}_{\{x\in\mathcal{A}\}}$$

B.4.5 UE Threats

$$UE\text{ Threats}\quad=\quad qv\left(x\right)\phi\left[\boldsymbol{J}\left(t_U\left(x\right)\right)+\sum_{i=1}^{n(x)}\boldsymbol{V}\left(t_U\left(x\right),i\right)-\Omega\left(x\right)\right]\cdot\mathbb{I}_{\{x\notin\mathcal{A}\}}$$

Using the definition of $\boldsymbol{\Omega}(t_U(x))$, we can re-write this term as

$$UE\text{ Threats}\quad=\quad qv\left(x\right)\phi\left[\boldsymbol{\Omega}\left(t_U\left(x\right)\right)-\Omega(x)\right]\cdot\mathbb{I}_{\{x\notin\mathcal{A}\}}$$

Now using our result in condition **C-UE** that $\boldsymbol{\Omega}\left(t_U\left(x\right)\right)=\Omega(x)$, we can conclude that

$$UE\text{ Threats}=0$$

B.4.6 EE Hires

$$EE\text{ Hires}=qv\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^E(x',i')}\left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right)+\sum_{i=1}^{n(x)}\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)-\Omega\left(x\right)\right]dH_n\left(x',i'\right)$$

Now by definition

$$\begin{aligned}\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)&=\boldsymbol{J}\left(h_E\left(x',i',x\right)\right)+\sum_{i=1}^{n(h_E(x',i',x))}\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)\\&=\left[\boldsymbol{J}\left(h_E\left(x',i',x\right)\right)+\sum_{i=1}^{n(x)}\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)\right]+\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)\end{aligned}$$

which can be re-arranged into

$$\boldsymbol{J}\left(h_E\left(x',i',x\right)\right)+\sum_{i=1}^{n(x)}\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)=\boldsymbol{\Omega}\left(h_E\left(x',i',x\right)\right)-\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)$$

Using this in the term in the square brackets

$$EE \text{ Hires} = qv\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^{E}\left(x',i'\right)}\left[\boldsymbol{\Omega}\left(h_{E}\left(x',i',x\right)\right)-\Omega\left(x\right)-\boldsymbol{V}\left(h_{E}\left(x',i',x\right),i\right)\right]dH_{n}\left(x',i'\right)$$

Under **C-EE**, the value going to the hired worker is $\boldsymbol{V}\left(h_E\left(x',i',x\right),i\right)=\Omega\left(x'\right)-\boldsymbol{\Omega}\left(q_E\left(x',i',x\right)\right)$. Substituting this in:

$$EE \text{ Hires} = qv\left(x\right)\left(1-\phi\right)\int_{x\in\mathcal{Q}^{E}\left(x',i'\right)}\left[\left[\boldsymbol{\Omega}\left(h_{E}\left(x',i',x\right)\right)-\Omega\left(x\right)\right]-\left[\Omega\left(x'\right)-\boldsymbol{\Omega}\left(q_{E}\left(x',i',x\right)\right)\right]\right]dH_{n}\left(x',i'\right)$$

B.4.7 EE Threats

$$EE \text{ Threats} \quad = \quad qv\left(x\right)\left(1-\phi\right)\int_{x\notin\mathcal{Q}^{E}\left(x',i'\right)}\left[\boldsymbol{J}\left(t_{E}\left(x',i',x\right)\right)+\sum_{i=1}^{n(x)}\boldsymbol{V}\left(t_{E}\left(x',i',x\right),i\right)-J\left(x\right)-\overline{V}\left(x\right)\right]dH_{n}\left(x',i'\right)$$

Using the definition of $\boldsymbol{\Omega}(t_E(x',i',x))$, we obtain

$$EE \text{ Threats} \quad = \quad qv\left(x\right)\left(1-\phi\right)\int_{x\notin\mathcal{Q}^{E}\left(x',i'\right)}\left[\boldsymbol{\Omega}\left(t_{E}\left(x',i',x\right)\right)-\Omega(x)\right]dH_{n}\left(x',i'\right)$$

Now using the result in condition C-RT that $\boldsymbol{\Omega}\left(t_E\left(x',i',x\right)\right)=\Omega(x)$, we obtain that

$$EE \text{ Threats} = 0$$

B.5 Reducing the state space

We have obtained the simplified recursion (4)-(10). Inspection of the system (4)-(10) reveals that the only payoff-relevant states are (z,n), and the details of the within-firm contractual structure do not affect allocations. Any extra information contained in x beyond (z,n) would be redundant given (z,n).

Therefore, it is straightforward to see that we may express the exit and separation decisions as

$$\begin{aligned}\boldsymbol{\Omega}(z,n) &= \mathbb{I}_{\{(z,n)\in\mathcal{E}\}}\left\{\vartheta+nU\right\}+\mathbb{I}_{\{(z,n)\in\mathcal{Q}^{U}\}}\left\{\boldsymbol{\Omega}(z,n-1)+U\right\}+\mathbb{I}_{\{(z,n)\notin\mathcal{Q}^{U}\cup\mathcal{E}\}}\Omega(z,n), \qquad (11)\\ \text{where}\quad \mathcal{E} &= \left\{n,z\middle|\vartheta+nU>\Omega(z,n)\right\},\\ \mathcal{Q}^{U} &= \left\{z,n\middle|\boldsymbol{\Omega}\left(z,n-1\right)+U>\Omega\left(z,n\right)\right\}.\end{aligned}$$

The first expression is the value of exit. A firm that does not exit, chooses whether to separate with a worker or not. If separating with a worker, the firm re-enters (11) with $\boldsymbol{\Omega}(z,n-1)$, having dispatched with a worker with value U, and again choosing whether to exit, fire another worker, or continue. Iterating on this procedure delivers

$$\boldsymbol{\Omega}(z,n)=\max\left\{\vartheta+nU\,,\max_{s\in[0,\ldots,n]}\Omega(z,n-s)+sU\right\}. \qquad (12)$$

Second, the post-exit/separation decision joint value is given by the Bellman equation

$$\begin{aligned}
\rho\Omega(z,n) &= \max_{v\geq 0} y(z,n) - c(v,n,z) \\
\text{Destruction} \quad &+ \quad \delta n\Big\{ \left(\boldsymbol{\Omega}(z,n-1)+U\right) - \Omega(z,n) \Big\} \\
UE \text{ Hire} \quad &+ \quad \phi q(\theta) v \cdot \mathbb{I}_{\{(z,n)\in\mathcal{A}\}} \cdot \Big\{ \boldsymbol{\Omega}(z,n+1) - \left(\Omega(z,n)+U\right) \Big\} \\
EE \text{ Hire} \quad &+ \quad (1-\phi)\, q(\theta) v \int_{(z,n)\in\mathcal{Q}^E(z',n')} \Big\{ \left[\boldsymbol{\Omega}(z,n+1) - \Omega(z,n)\right] - \left[\Omega(z',n') - \boldsymbol{\Omega}(z',n'-1)\right] \Big\} dH_n(z',n') \\
\text{Shock} \quad &+ \quad \Gamma_z[\boldsymbol{\Omega},\Omega](z,n), \\
\text{where} \quad \mathcal{A} &= \left\{ z,n \middle| \boldsymbol{\Omega}(z,n+1) \geq \Omega(z,n) + U \right\}, \\
\mathcal{Q}^E(z',n') &= \left\{ z,n \middle| \boldsymbol{\Omega}(z,n+1) - \Omega(z,n) \geq \Omega(z',n') - \boldsymbol{\Omega}(z',n'-1) \right\}.
\end{aligned}$$

Finally, firms enter if and only if

$$\int \boldsymbol{\Omega}(z,0)\, d\Pi_0(z) \geq c_e. \tag{13}$$

This condition pins down the entry rate per unit of time.[4] Details of the proof are available upon request.

B.6 Continuous workforce limit

Up to this point the economy has featured a continuum of firms, but an integer-valued workforce. We now take the continuous workforce limit by defining the 'size' of a worker—which is 1 in the integer case—and taking the limit as this approaches zero. Specifically, denote the "size" of a worker by Δ, such that $n = N\Delta$ where N is the old integer number of workers. Now define $\Omega^{\Delta}(z,n) := \Omega(z,n/\Delta)$, and likewise define $y^{\Delta}(z,n) := y(z,n/\Delta)$ and $c^{\Delta}(v,n,z) := c(v/\Delta, n/\Delta, z)$. We also define $b^{\Delta} := b/\Delta$ and $\vartheta^{\Delta} := \vartheta/\Delta$. These imply, for example, that $\Omega(z,N) = \Omega^{\Delta}(z,N\Delta)$. Substituting these terms into (12) and (13), and taking the limit $\Delta \to 0$, while holding $n = N\Delta$ fixed, we would obtain a version of (14) in which all functions have the Δ super-script notation. We also specialize the productivity to a diffusion process $dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$.

The result is the joint value representation of Section 3: a Hamilton-Jacobi-Bellman (HJB) equation for the joint value *conditional on the firm and its workers operating*:

$$\begin{aligned}
\rho\Omega(z,n) = \max_{v\geq 0} \quad & y(z,n) - c(v,n,z) \\
\text{Destruction} \quad & -\delta n[\Omega_n(z,n) - U] \\
UE \text{ Hire} \quad & +\phi q(\theta) v\,[\Omega_n(z,n) - U] \\
EE \text{ Hire} \quad & +(1-\phi) q(\theta) v \int \max\Big\{ \Omega_n(z,n) - \Omega_n(n',z'),\, 0 \Big\} dH_n(z',n') \\
\text{Shock} \quad & +\mu(z)\Omega_z(z,n) + \frac{\sigma(z)^2}{2}\Omega_{zz}(z,n).
\end{aligned} \tag{14}$$

Boundary conditions for the firm and its workers operating require the state to be interior to the exit and separation boundaries:

$$\begin{aligned}
\text{Exit boundary:} \quad & \Omega(z,n) \geq \vartheta + nU, \\
\text{Layoff boundary:} \quad & \Omega_n(z,n) \geq U
\end{aligned}$$

Note the absence of $\boldsymbol{\Omega}$ terms. Since the value we track is that of a hiring firm subject to boundary conditions, then

[4] Recall that $J_0 = -c_e + \int \boldsymbol{J}(x_0) d\Pi(z_0)$. Given $\Omega(z_0,0) = J(z_0,0)$, we have $J_0 = -c_e + \int \boldsymbol{\Omega}(z_0,0) d\Pi(z_0)$. Free-entry implies $J_0 = 0$, which delivers (13).

$\boldsymbol{\Omega} = \Omega$. This admits the simplification of 'Shock' terms we noted when discussing (1).

We proceed in three steps:

(B.6.1) Define worker size and the renormalization

(B.6.2) Take the limit as worker size goes to zero

(B.6.3) Introduce a continuous productivity process.

B.6.1 Define worker size and the renormalization

We denote the "size" of a worker by Δ. That is, we currently have an integer work-force $n \in \{1, 2, 3, \dots\}$. We now consider $m \in \{\Delta, 2\Delta, 3\Delta, \dots\}$. So then $n = m/\Delta$. We use this to make the following normalizations:

$$\omega(z, m) = \Omega\left(\frac{m}{\Delta}, z\right) \qquad \mathcal{Y}(z, m) = y\left(\frac{m}{\Delta}, z\right) \qquad \mathcal{C}(z, m) = c\left(\frac{v}{\Delta}, \frac{m}{\Delta}, z\right)$$

These definition imply

$$\Omega(z, n) = \omega(n\Delta, z) \qquad y(z, n) = \mathcal{Y}(n\Delta, z) \qquad c(v, z, n) = \mathcal{C}(v\Delta, n\Delta, z)$$

In addition, the value of unemployment solves $\rho U = b$. Define

$$\mathcal{U} = \frac{b}{\rho\Delta} = \frac{U}{\Delta}$$

and

$$\theta = \frac{\vartheta}{\Delta}$$

Substituting these definitions into the Bellman equation, we obtain

$$\begin{aligned}
\rho\omega\left(n\Delta, z\right) = \max_{v\Delta \geq 0} \quad & \mathcal{Y}\left(n\Delta, z\right) - \mathcal{C}\left(v\Delta, n\Delta, z\right) \\
\text{Destructions} \quad & -\delta n\Delta\left[\frac{\omega\left(n\Delta, z\right) - \boldsymbol{\omega}\left(n\Delta - \Delta, z\right)}{\Delta} - \mathcal{U}\right] \\
UE \text{ Hires} \quad & +qv\Delta\phi\left[\frac{\boldsymbol{\omega}\left(n\Delta + \Delta, z\right) - \omega\left(n\Delta, z\right)}{\Delta} - \mathcal{U}\right] \cdot \mathbb{I}_{\{(n\Delta, z) \in \mathcal{A}\}} \\
EE \text{ Hires} \quad & +qv\Delta\left(1 - \phi\right)\int_{(n\Delta, z) \in \mathcal{Q}^E(n'\Delta, z')}\left[\frac{\boldsymbol{\omega}\left(n\Delta + \Delta, z\right) - \omega\left(n\Delta, z\right)}{\Delta} - \frac{\omega\left(n'\Delta, z'\right) - \boldsymbol{\omega}\left(n'\Delta - \Delta, z'\right)}{\Delta}\right] d\widetilde{H_n}\left(n'\Delta, z'\right) \\
\text{Shocks} \quad & +\Gamma_z\left[\boldsymbol{\omega}, \omega\right]\left(n\Delta, z\right)
\end{aligned}$$

with the set definitions

$$\mathcal{E} = \left\{ n\Delta, z \middle| \max_{k\Delta \in \{0,\dots,n\Delta\}} \omega(k\Delta, z) + (n\Delta - k\Delta)\mathcal{U} < \theta + n\Delta\mathcal{U} \right\}$$

$$\mathcal{A} = \left\{ n\Delta, z \middle| \frac{\boldsymbol{\omega}\left(n\Delta + \Delta, z\right) - \omega\left(n\Delta, z\right)}{\Delta} \geq \mathcal{U} \right\}$$

$$\mathcal{Q}^U = \left\{ n\Delta, z \middle| \frac{\omega\left(n\Delta, z\right) - \boldsymbol{\omega}\left(n\Delta - \Delta, z\right)}{\Delta} \leq \mathcal{U} \right\}$$

$$\mathcal{Q}^E\left(n'\Delta, z'\right) = \left\{ n\Delta, z \middle| \frac{\boldsymbol{\omega}\left(n\Delta + \Delta, z\right) - \omega\left(n\Delta, z\right)}{\Delta} \geq \frac{\omega\left(n'\Delta, z'\right) - \boldsymbol{\omega}\left(n'\Delta - \Delta, z'\right)}{\Delta} \right\}$$

and the definition:

$$\boldsymbol{\omega}(n\Delta, z) \quad = \quad \max\left\{ \max_{k\Delta \in \{0,\dots,n\Delta\}} \omega(k\Delta, z) + (n\Delta - k\Delta)\mathcal{U}\ ,\ \theta + n\Delta\mathcal{U} \right\}$$

B.6.2 Continuous limit as worker size goes to zero

Now we take the limit $\Delta \to 0$, holding $m = n\Delta$ fixed. We note $\widehat{v} = \lim_{\Delta\to 0} v\Delta$. We see derivatives appear. We denote $\omega_m(z, m) = \frac{\partial\omega}{\partial m}(z, m)$.

First, we note that the following limit obtains:

$$\boldsymbol{\omega}(z, m) \quad = \quad \max\left\{ \max_{k\in[0,m]} \omega(k, z) + (m - k)\mathcal{U}\ ,\ \theta + m\Delta\mathcal{U} \right\}$$

In particular, the exit set limits to

$$\mathcal{E} = \left\{ z, m \middle| \max_{k\in[0,m]} \omega(k, z) + (m - k)\mathcal{U} < \theta + m\mathcal{U} \right\}$$

In equilibrium, the $\boldsymbol{\omega}(z, m)$ terns on the right-hand-side of the Bellman equation are the result of endogenous quits, layoffs and hires. Because our continuous time assumption has been made *before* we take the limit to a continuous workforce limit, we need only consider those changes in the workforce one at a time. Hence, for any $(z, m) \in$ Interior$(\mathcal{E}^c \cap \mathcal{A})$, the *interior* of the continuation set, there is always $\overline{\Delta} > 0$: such that for any $\Delta \leq \overline{\Delta}$:

$$\boldsymbol{\omega}(m \pm \Delta, z) = \omega(m \pm \Delta, z)$$

Using this observation in the Bellman equation, we can obtain derivatives on the right-hand-side. We obtain, for pairs (z, n) in the interior of the continuation set $(z, n) \in$ Interior$(\mathcal{E}^c \cap \mathcal{A})$:

$$\begin{aligned}
\rho\omega\left(z, m\right) = \max_{\widehat{v}\geq 0} \quad & \mathcal{Y}\left(z, m\right) - \mathcal{C}\left(\widehat{v}, z, m\right) \\
\text{Destructions} \quad & -\delta m[\omega_m(z, m) - \mathcal{U}] \\
UE \text{ Hires} \quad & +q\widehat{v}\phi\left[\omega_m(z, m) - \mathcal{U}\right] \cdot \mathbb{I}_{\{(z,m)\in\mathcal{A}\}} \\
EE \text{ Hires} \quad & +q\widehat{v}\left(1 - \phi\right) \int_{(z,m)\in\mathcal{Q}^E(m',z')} \left[\omega_m(z, m) - \omega_m(m', z')\right] d\widetilde{H_n}\left(m', z'\right) \\
\text{Shocks} \quad & +\Gamma_z\left[\boldsymbol{\omega}, \omega\right](z, n)
\end{aligned}$$

with the set definitions

$$\mathcal{E} = \left\{ z, m \middle| \max_{k \in [0,m]} \omega(k,z) + (n-k)\mathcal{U} < \theta + m\mathcal{U} \right\} \qquad \mathcal{A} = \left\{ z, m \middle| \omega_m(z,m) \geq \mathcal{U} \right\}$$

$$\mathcal{Q}^U = \left\{ z, m \middle| \omega_m(z,m) \leq \mathcal{U} \right\} = \overline{A} \text{ , the complement of } \mathcal{A}$$

$$\mathcal{Q}^E \left(z', m'\right) = \left\{ z, m \middle| \omega_m(z,m) - \omega_m(m', z') \geq 0 \right\}$$

and the definition

$$\boldsymbol{\omega}(z,m) = \max \left\{ \max_{k \in [0,m]} \omega(k,z) + (m-k)\mathcal{U} \text{ , } \theta + m\mathcal{U} \right\}$$

Note that now, the only place where $\boldsymbol{\omega}$ enters in the Bellman equation is the contribution of shocks. To replace it with ω, we need to apply the same argument to z as the one we applied to n. We thus need to specialize to a continuous productivity process.

B.6.3 Continuous productivity process

We now specialize to a continuous productivity process, as this makes the formulation of the problem very economical. It allows to simplify the contribution of productivity shocks and get rid of the remaining "bold" notation. We suppose that productivity follows a diffusion process:

$$dz_t = \mu(z_t)dt + \sigma(z_t)dW_t$$

In this case, for any (z, m) in the interior of the continuation set, productivity shocks in the interval $[t, t + dt]$ cannot move the firm towards a region in which it would endogenously separate or exit, when dt is small enough. In this case, we can write the following, where we have also replaced the $\mathcal{Q}^E$ set with the max operator:

$$\begin{aligned}
\rho\omega(z,m) = \max_{v \geq 0} \quad & \mathcal{Y}(z,m) - \mathcal{C}(v,z,m) \\
\text{Destructions} \quad & -\delta m[\omega_m(z,m) - \mathcal{U}] \\
UE \text{ Hires} \quad & +qv\phi\left[\omega_m(z,m) - \mathcal{U}\right] \\
EE \text{ Hires} \quad & +qv(1-\phi)\int \max\left\{ \omega_m(z,m) - \omega_m(z',m') \text{ , } 0 \right\} d\widetilde{H_n}\left(m', z'\right) \\
\text{Shocks} \quad & +\mu(z)\omega_z(z,m) + \frac{\sigma(z)^2}{2}\omega_{zz}(z,m) \\
\text{s.t.} \quad & \\
\text{No Exit} \quad & \omega(z,m) \geq \theta + m\mathcal{U} \\
\text{No Separations} \quad & \omega_m(z,m) \geq \mathcal{U}
\end{aligned}$$

To make the notation more comparable, we slightly abuse notation and use the same letters as before, but now for the continuous workforce case. We obtain finally:

$$
\begin{aligned}
\rho\Omega\left(z,n\right) = \max_{v\geq 0} \quad & y\left(z,n\right) - c\left(v,z,n\right) \\
\text{Destructions} \quad & -\delta n[\Omega_n(z,n) - U] \\
UE \text{ Hires} \quad & +qv\phi\left[\Omega_n(z,n) - U\right] \\
EE \text{ Hires} \quad & +qv\left(1-\phi\right)\int \max\left[\Omega_n(z,n) - \Omega_n(z',n')\ ,\ 0\right] d\widetilde{H_n}\left(z',n'\right) \\
\text{Shocks} \quad & +\mu(z)\Omega_z(z,n) + \frac{\sigma(z)^2}{2}\Omega_{zz}(z,n) \\
\text{s.t.} & \\
\text{No Exit} \quad & \Omega(z,n) \geq \vartheta + nU \\
\text{No Separations} \quad & \Omega_n(z,n) \geq U
\end{aligned}
$$

When the coalition hits $\Omega_n(z,n) = U$, it endogenous separates worker to stay on that frontier. It exits when it hits the frontier $\Omega(z,n) = \vartheta + nU$.

In addition to these "value-pasting" boundary conditions, optimality implies necessary "smooth-pasting" boundary conditions (see Stokey 2009): $\Omega_z(z,n) = 0$ if the firm actually exits at (z,n) following productivity shocks, and $\Omega_n(z,n) = 0$ if the firm actually exits at (z,n) following changes in size. These are necessary and sufficient for the definition of our problem (Brekke and Øksendal 1991). Its general formulation terms of optimal switching between three regimes (operation, layoffs, exit) on the entire positive quadrant, can be made as a system of Hamilton-Jacobi-Bellman-Variational-Inequality (see Pham 2009), which we present here for completeness :

$$
\begin{aligned}
\max\Bigg\{ & -\rho\Omega\left(z,n\right) + \max_{v\geq 0} -\delta n[\Omega_n(z,n) - U] + qv\phi\left[\Omega_n(z,n) - U\right] \\
& + qv\left(1-\phi\right)\int \max\left[\Omega_n(z,n) - \Omega_n(z',n')\ ,\ 0\right] d\widetilde{H_n}\left(z',n'\right) + \mu(z)\Omega_z(z,n) + \frac{\sigma(z)^2}{2}\Omega_{zz}(z,n)\ ; \\
& \vartheta + nU - \Omega(z,n)\ ;\ \max_{k\in[0,n]} \Omega(z,k) + (n-k)U - \Omega(z,n) \Bigg\} = 0 \quad , \quad \forall (z,n) \in \mathbb{R}^2_+
\end{aligned}
$$

C Computational details

C.1 Numerical Solution to Surplus HJB Equation

C.1.1 Simplifying the Bellman equation

- Recall that the stochastic process for z_t is

$$dz_t = \mu(z_t)\,dt + \sigma(z_t)\,dW_t$$

- We want to solve:

$$\begin{aligned}\widetilde{\rho}S(z,n) = \max_{v\geq 0}\, & y(z,n) - nb - c_f - \delta n S_n(z,n) \\ & \underbrace{+\, q\left[\phi S_n(z,n) + (1-\phi)\int_0^{S_n(z,n)} \left[S_n(z,n) - S_n'\right] dH_n\left(S_n'\right)\right]}_{\text{Expected benefit per vacancy} := \mathcal{H}(S_n)} v - c(v,n) \\ & + \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2}S_{zz}(z,n)\end{aligned}$$

where $\widetilde{\rho} = (\rho + \delta_x)$ to account for exogneous exit at rate δ_x, which we use in the quantitative model.

- The expected benefit per vacancy, $\mathcal{H}(S_n)$ is a function only of marginal surplus S_n, and integrating by parts is:

$$\mathcal{H}(S_n) = q\left[\phi S_n + (1-\phi)\widehat{H}_n(S_n)\right]$$

where

$$\widehat{H}_n(S_n) = \int_0^{S_n} H_n(s)ds \quad , \quad H_n(S_n) = \int_0^{S_n} h_n(s)\,ds \quad , \quad h_n(S_n) = \frac{n(S_n)}{\mathtt{n}} h(S_n)$$

- As in our quantitative exercise, let $c(v,n) = \frac{\bar{c}}{1+\gamma}\left(\frac{v}{n}\right)^{\gamma} v$.
- The first order condition for vacancies is then as follows, with associated vacancy rate:

$$\begin{aligned}\mathcal{H}(S_n) &= \bar{c}\left(\frac{v}{n}\right)^{\gamma} \\ \frac{v(z,n)}{n} &= \frac{1}{\bar{c}^{1/\gamma}}\mathcal{H}(S_n(z,n))^{\frac{1}{\gamma}}.\end{aligned}$$

- The terms that depend on vacancies in the Bellman equation can therefore be simplified:

$$\begin{aligned}\mathcal{H}(S_n(z,n))\,v - c(v,n) &= \xi \mathcal{H}(S_n(z,n))^{\frac{1+\gamma}{\gamma}}\, n\,, \\ \xi &:= \frac{1}{\bar{c}^{1/\gamma}}\frac{\gamma}{1+\gamma}.\end{aligned}$$

- Substituting this back into the Bellman equation, and re-arranging we have flow payoffs, terms that depend on the drift of n and terms that depend on the dynamics of z:

$$\begin{aligned}\widetilde{\rho}S(z,n) = y(z,n) - nb - c_f & \\ + \left[\xi\frac{\mathcal{H}(S_n(z,n))^{\frac{1+\gamma}{\gamma}}}{S_n(z,n)} - \delta\right] n S_n(z,n) & \\ + \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2}S_{zz}(z,n). &\end{aligned} \tag{15}$$

- This is subject to boundary conditions and the previous definitions:

$$
\begin{aligned}
S(z,n) &\geq 0 \\
S_n(z,n) &\geq 0 \\
\mathcal{H}(S_n(z,n)) &= q\left[\phi S_n(z,n) + (1-\phi)\widehat{H}_n(S_n(z,n))\right] \\
\widehat{H}_n(S_n(z,n)) &= \int_0^{S_n(z,n)} H_n(s)ds
\end{aligned}
$$

C.1.2 Change of variables

- Define the following objects

$$
\begin{aligned}
\mu_n(z,n) &:= \xi \frac{\mathcal{H}(S_n(z,n))^{\frac{1+\gamma}{\gamma}}}{S_n(z,n)} - \delta \\
\pi(z,n) &:= y(z,n) - nb - c_f
\end{aligned}
$$

- Up to this point, the stochastic process for z_t has been a general geometric Brownian motion with drift and volatility $\mu(z_t)$and $\sigma(z_t)$, respectively:

$$
dz_t = \mu(z_t)\,dt + \sigma(z_t)\,dW_t
$$

- In the quantitative model, we consider a random walk in logs:

$$
d\log z_t = \mu dt + \sigma dW_t
$$

- Ito's Lemma implies that

$$
dz_t = \underbrace{\left[\mu + \frac{\sigma^2}{2}\right] z_t}_{\mu(z_t)}\,dt + \underbrace{\sigma z_t}_{\sigma(z_t)}\,dW_t
$$

- Substituting these into the Bellman equation

$$
\begin{aligned}
\widetilde{\rho} S(z,n) &= \pi(z,n) + \mu_n(z,n)\,nS_n(z,n) + \mu(z)S_z(z,n) + \frac{\sigma^2(z)}{2}S_{zz}(z,n) \\
\widetilde{\rho} S(z,n) &= \pi(z,n) + \mu_n(z,n)\,nS_n(z,n) + \left[\mu + \frac{\sigma^2}{2}\right] zS_z(z,n) + \frac{\sigma^2}{2}z^2 S_{zz}(z,n)
\end{aligned}
$$

- Now consider a change of variables. Let $\widetilde{z} = \log z$, $\widetilde{n} = \log n$. Now define $\widetilde{S}(\widetilde{z},\widetilde{n}) = S\left(e^{\widetilde{z}}, e^{\widetilde{n}}\right) = S(z,n)$, $\widetilde{\pi}(\widetilde{z},\widetilde{n}) = \pi\left(e^{\widetilde{z}}, e^{\widetilde{n}}\right)$, and $\widetilde{\mu}_n(\widetilde{z},\widetilde{n}) = \mu_n\left(e^{\widetilde{z}}, e^{\widetilde{n}}\right)$
- Note that

$$
\begin{aligned}
\widetilde{\pi}(\widetilde{z},\widetilde{n}) &= y\left(e^{\widetilde{z}}, e^{\widetilde{n}}\right) - e^{\widetilde{n}}b - c_f = \widetilde{y}(\widetilde{z},\widetilde{n}) - e^{\widetilde{n}}b - c_f \\
\widetilde{y}(\widetilde{z},\widetilde{n}) &= \left(e^{\widetilde{z}}\right) \times \left(e^{\alpha\widetilde{n}}\right)
\end{aligned}
$$

- Applying the chain rule to $S(z,n) = \widetilde{S}(\log z, \log n)$, and re-arranging:

$$S_n(z,n)\,n = \widetilde{S}_{\widetilde{n}}(\widetilde{z},\widetilde{n})$$
$$S_z(z,n)\,z = \widetilde{S}_{\widetilde{z}}(\widetilde{z},\widetilde{n})$$
$$z^2 S_{zz}(z,n) = \widetilde{S}_{\widetilde{z}\widetilde{z}}(\widetilde{z},\widetilde{n}) - \widetilde{S}_{\widetilde{z}}(\widetilde{z},\widetilde{n})$$

- Substituting these into the Bellman equation

$$\widetilde{\rho}\widetilde{S}(\widetilde{z},\widetilde{n}) = \widetilde{\pi}(\widetilde{z},\widetilde{n}) + \mu_n\left(S_n\left(e^{\widetilde{z}}, e^{\widetilde{n}}\right)\right)\widetilde{S}_{\widetilde{n}}(\widetilde{z},\widetilde{n}) + \mu\widetilde{S}_{\widetilde{z}}(\widetilde{z},\widetilde{n}) + \frac{\sigma^2}{2}\widetilde{S}_{\widetilde{z}\widetilde{z}}(\widetilde{z},\widetilde{n}) \tag{16}$$

- The boundary conditions are the same, since $S(z,n) = \widetilde{S}(\widetilde{z},\widetilde{n})$, and $S_n(z,n) \geq 0$, which since $n \geq 0$, is true if and only if $S_n(z,n)\,n \geq 0$, which is equivalent to $\widetilde{S}_{\widetilde{n}}(\widetilde{z},\widetilde{n}) \geq 0$.

C.1.3 Implicit method

- We solve (16) using an implicit method.
- Let Δ denote step-size and τ the iteration of the algorithm.
- Then given $\widetilde{S}^{\tau-1}(\widetilde{z},\widetilde{n})$, the implicit method gives an update:

$$\frac{1}{\Delta}\left[\widetilde{S}^{\tau}(\widetilde{z},\widetilde{n}) - \widetilde{S}^{\tau-1}(\widetilde{z},\widetilde{n})\right] + \widetilde{\rho}S^{\tau}(\widetilde{z},\widetilde{n}) = \widetilde{\pi}(\widetilde{z},\widetilde{n}) + \mu_n\left(S_n\left(e^{\widetilde{z}}, e^{\widetilde{n}}\right)\right)\widetilde{S}^{\tau}_{\widetilde{n}}(\widetilde{z},\widetilde{n}) + \mu\widetilde{S}^{\tau}_{\widetilde{z}}(\widetilde{z},\widetilde{n}) + \frac{\sigma^2}{2}\widetilde{S}^{\tau}_{\widetilde{z}\widetilde{z}}(\widetilde{z},\widetilde{n})$$

- Rearranging this expression:

$$\left(\frac{1}{\Delta} + \widetilde{\rho}\right)\widetilde{S}^{\tau}(\widetilde{z},\widetilde{n}) - \mu_n\left(S_n\left(e^{\widetilde{z}}, e^{\widetilde{n}}\right)\right)\widetilde{S}^{\tau}_{\widetilde{n}}(\widetilde{z},\widetilde{n}) - \mu\widetilde{S}^{\tau}_{\widetilde{z}}(\widetilde{z},\widetilde{n}) - \frac{\sigma^2}{2}\widetilde{S}^{\tau}_{\widetilde{z}\widetilde{z}}(\widetilde{z},\widetilde{n}) = \widetilde{\pi}(\widetilde{z},\widetilde{n}) + \frac{1}{\Delta}\widetilde{S}^{\tau-1}(\widetilde{z},\widetilde{n}) \tag{17}$$

- We now discretize $\widetilde{n}$ on an evenly spaced $N_{\widetilde{n}} \times 1$ grid, $\widetilde{\boldsymbol{n}} = (\widetilde{n}_0, \widetilde{n}_0 + \Delta_{\widetilde{n}}, \widetilde{n}_0 + 2\Delta_{\widetilde{n}}, \widetilde{n}_0 + (N_{\widetilde{n}} - 1)\Delta_{\widetilde{n}})$, and $\widetilde{z}$ on an evenly spaced $N_{\widetilde{z}} \times 1$ grid, $\widetilde{\boldsymbol{z}} = (\widetilde{z}_0, \widetilde{z}_0 + \Delta_{\widetilde{z}}, \widetilde{z}_0 + 2\Delta_{\widetilde{z}}, \widetilde{z}_0 + (N_{\widetilde{z}} - 1)\Delta_{\widetilde{z}})$.
- Stack these according to:

$$\widetilde{\boldsymbol{s}} = \left[\boldsymbol{i}_{N_{\widetilde{n}}} \otimes \widetilde{\boldsymbol{z}}, \widetilde{\boldsymbol{n}} \otimes \boldsymbol{i}_{N_{\widetilde{z}}}\right] = \begin{pmatrix} \widetilde{z}_1, \widetilde{n}_1 \\ \widetilde{z}_2, \widetilde{n}_1 \\ \vdots \\ \widetilde{z}_{N_{\widetilde{z}}}, \widetilde{n}_1 \\ \vdots \\ \widetilde{z}_1, \widetilde{n}_{N_{\widetilde{n}}} \\ \vdots \\ \widetilde{z}_{N_{\widetilde{z}}}, \widetilde{n}_{N_{\widetilde{n}}} \end{pmatrix}.$$

- Discretized, we can write (17) as $N_{\widetilde{z}} \times N_{\widetilde{n}}$ equations:

$$\left(\frac{1}{\Delta} + \widetilde{\rho}\right)\widetilde{S}^{\tau} - \mu_n\widetilde{S}^{\tau}_{\widetilde{n}} - \mu\widetilde{S}^{\tau}_{\widetilde{z}} - \frac{\sigma^2}{2}\widetilde{S}^{\tau}_{\widetilde{z}\widetilde{z}} = \widetilde{\pi} + \frac{1}{\Delta}\widetilde{S}^{\tau-1} \tag{18}$$

- Let $D_{\widetilde{n}}$ be the $(N_{\widetilde{z}} \times N_{\widetilde{n}}) \times (N_{\widetilde{z}} \times N_{\widetilde{n}})$ square matrix that, when pre-multiplying $\widetilde{S}^{\tau}$, gives an approximation of

$\widetilde{S}^{\tau}_{\widetilde{n}}$. Analogously, define $D_{\widetilde{z}}$ and $D_{\widetilde{z}\widetilde{z}}$:

$$\widetilde{S}^{\tau}_{\widetilde{n}} = D_{\tilde{n}} \widetilde{S}^{\tau}$$
$$\widetilde{S}^{\tau}_{\widetilde{z}} = D_{\widetilde{z}} \widetilde{S}^{\tau}$$
$$\widetilde{S}^{\tau}_{\widetilde{z}\widetilde{z}} = D_{\widetilde{z}\widetilde{z}} \widetilde{S}^{\tau}$$

- Using these we can write (18) as

$$\left[\left(\frac{1}{\Delta} + \widetilde{\rho} \right) - \widetilde{\mu}_n D_{\tilde{n}} - \left(\mu D_{\widetilde{z}} + \frac{\sigma^2}{2} D_{\widetilde{z}\widetilde{z}} \right) \right] \widetilde{S}^{\tau} = \widetilde{\pi} + \frac{1}{\Delta} \widetilde{S}^{\tau-1}$$

- Define

$$\mathcal{N}^v = \widetilde{\mu}_n D_{\tilde{n}}$$
$$\mathcal{Z}^v = \mu D_{\widetilde{z}} + \frac{\sigma^2}{2} D_{\widetilde{z}\widetilde{z}}$$

- Then we have

$$\left[\left(\frac{1}{\Delta} + \widetilde{\rho} \right) - \mathcal{N}^v - \mathcal{Z}^v \right] \widetilde{S}^{\tau} = \widetilde{\pi} + \frac{1}{\Delta} \widetilde{S}^{\tau-1}$$

- Therefore the implicit method works by updating S^{τ} through

$$\mathbf{B} S^{\tau} = \Pi + \frac{1}{\Delta} S^{\tau-1}$$
$$\mathbf{B} = \left(\frac{1}{\Delta} + \widetilde{\rho} \right) - \mathcal{N}^v - \mathcal{Z}^v \tag{19}$$

C.1.4 Derivative matrices

- To compute the derivative matrices $D_{\tilde{n}}$, $D_{\widetilde{z}}$, and $D_{\widetilde{z}\widetilde{z}}$, we follow an upwind scheme.
- That is, we use a forward approximation when the drift of the state variable is positive, and a backward approximation when the drift of the state is negative.
- In the simple case of $D_{\widetilde{z}}$, since our estimation delivers $\mu < 0$, $D_{\widetilde{z}}$ is built considering a backward difference such that, for example, the derivative at point $S_z(z_3, n_2)$ is computed as

$$\widetilde{S}_z(\widetilde{z}_3, \widetilde{n}_2) = \frac{\widetilde{S}_z(\widetilde{z}_3, \widetilde{n}_2) - \widetilde{S}_z(\widetilde{z}_2, \widetilde{n}_2)}{\Delta_{\widetilde{z}}}.$$

- This requires that $D_{\widetilde{z}}$ is as follows, where $\boldsymbol{I}_{N_{\tilde{n}}}$ is an $N_{\tilde{n}} \times N_{\tilde{n}}$ identity matrix:

$$D_{\widetilde{z}} = \begin{pmatrix} -1/\Delta_{\widetilde{z}} & 1/\Delta_{\widetilde{z}} & 0 & \cdots & \cdots & 0 \\ -1/\Delta_{\widetilde{z}} & 1/\Delta_{\widetilde{z}} & 0 & \ddots & \ddots & \vdots \\ 0 & -1/\Delta_{\widetilde{z}} & 1/\Delta_{\widetilde{z}} & 0 & \ddots & \vdots \\ \vdots & 0 & -1/\Delta_{\widetilde{z}} & 1/\Delta_{\widetilde{z}} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & -1/\Delta_{\widetilde{z}} & 1/\Delta_{\widetilde{z}} \end{pmatrix}_{N_{\widetilde{z}} \times N_{\widetilde{z}}} \otimes \boldsymbol{I}_{N_{\tilde{n}}}$$

which gives a backward approximation for any i except for $i = 1$ in which case it uses a forward approximation.

- For the case of the second derivative with respect to z, we use a central approximation by building the following matrix[5]

$$D_{\tilde{z}\tilde{z}} = \begin{pmatrix} -1/\Delta_{\tilde{z}} & 1/\Delta_{\tilde{z}} & 0 & \cdots & \cdots & 0 \\ 1/\Delta_{\tilde{z}}^2 & -2/\Delta_{\tilde{z}}^2 & 1/\Delta_{\tilde{z}}^2 & 0 & \ddots & \vdots \\ 0 & 1/\Delta_{\tilde{z}}^2 & -2/\Delta_{\tilde{z}}^2 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 1/\Delta_{\tilde{z}}^2 & 0 \\ \vdots & \ddots & \ddots & 1/\Delta_{\tilde{z}}^2 & -2/\Delta_{\tilde{z}}^2 & 1/\Delta_{\tilde{z}}^2 \\ 0 & \cdots & \cdots & 0 & 1/\Delta_{\tilde{z}}^2 & -1/\Delta_{\tilde{z}}^2 \end{pmatrix}_{N_{\tilde{z}} \times N_{\tilde{z}}} \otimes \boldsymbol{I}_{N_{\tilde{n}}}$$

- Finally, care must be taken in the case of $D_{\tilde{n}}$ since the drift of n endogenously depends on $(\tilde{z}, \tilde{n})$, and thus the upwind scheme dictates that the direction of the finite difference depends on $(\tilde{z}, \tilde{n})$. We construct both $D_{\tilde{n}}^f$ that considers a forward approximation, $D_{\tilde{n}}^b$ for a backward approximation, and use the former for positive values of $\tilde{\mu}_n(\tilde{z}, \tilde{n})$ and the latter for negative values,

F.2 Numerical Solution to Kolmogorov Forward Equation

- Substituting these into the Bellman equation

$$\tilde{\rho}\tilde{S}(\tilde{z}, \tilde{n}) = \tilde{\pi}(\tilde{z}, \tilde{n}) + \tilde{\mu}_n(\tilde{z}, \tilde{n})\tilde{S}_{\tilde{n}}(\tilde{z}, \tilde{n}) + \mu\tilde{S}_{\tilde{z}}(\tilde{z}, \tilde{n}) + \frac{\sigma^2}{2}\tilde{S}_{\tilde{z}\tilde{z}}(\tilde{z}, \tilde{n}) \tag{20}$$

- We continue to work in the transformed variables.
- Note that

$$\frac{dn/n}{dt} = \frac{d\log n}{dt} = \frac{d\tilde{n}}{dt}$$

- Let $h(\tilde{z}, \tilde{n})$ be the stationary distribution of firms in the economy.
- Recall $\tilde{S}(\tilde{z}, \tilde{n}) = S(z, n)$, so operating firms have $\tilde{S}(\tilde{z}, \tilde{n}) \geq 0$.
- Then $h(\tilde{z}, \tilde{n})$ satisfies

$$\begin{aligned} 0 = &-\frac{\partial}{\partial\tilde{n}}\left(\left.\frac{d\tilde{n}(\tilde{z}, \tilde{n})}{dt}\right|_{\tilde{S}(\tilde{z},\tilde{n})\geq 0} h(\tilde{z}, \tilde{n})\right) + \left(\left.\frac{d\tilde{n}(\tilde{z}, \tilde{n})}{dt}\right|_{\tilde{S}(\tilde{z},\tilde{n})<0} h(\tilde{z}, \tilde{n})\right) \\ &- \mu h_{\tilde{z}}(\tilde{z}, \tilde{n}) + \frac{\sigma^2}{2}h_{\tilde{z}\tilde{z}}(\tilde{z}, \tilde{n}) - \delta_x h(\tilde{z}, \tilde{n}) + \mathtt{m}_0\pi_0(\tilde{z})\Delta(\tilde{n}) \end{aligned}$$

  Note that firm exit from negative surplus is not treated as part of the drift in n, hence two $\frac{d\tilde{n}}{dt}$ terms: firm exit takes mass away from certain states without returning it anywhere in particular, while layoffs shift mass from one state to another.

- We can vectorize this in the same way as above, and obtain

$$0 = -D_{\tilde{n}}\dot{\tilde{n}}h - \mu D_{\tilde{z}}h + \frac{\sigma^2}{2}D_{\tilde{z}\tilde{z}}h - \mathcal{X}h - \delta_x h + h_0$$

[5] For any j the (i, j) entry of $\tilde{S}_{\tilde{z}\tilde{z}} = D_{\tilde{z}}\tilde{S}$ reads $\frac{[\tilde{S}(\tilde{z}_{i+1},\tilde{n}_j) - \tilde{S}(z_i,\tilde{n}_j)] - [\tilde{S}(\tilde{z}_i,\tilde{n}_j) - \tilde{S}(\tilde{z}_{i-1},\tilde{n}_j)]}{\Delta_{\tilde{z}}^2}$ for any i except for $i = 1$ in which case it reads $\frac{\tilde{S}(\tilde{z}_2,\tilde{n}_j) - \tilde{S}(\tilde{z}_1,\tilde{n}_j)}{\Delta_{\tilde{z}}^2}$ and for $i = N_{\tilde{z}}$ in which case it reads $\frac{\tilde{S}(\tilde{z}_{N_z},\tilde{n}_j) - \tilde{S}(\tilde{z}_{N_{\tilde{z}}-1},\tilde{n}_j)}{\Delta_{\tilde{z}}^2}$.

- Where (i) h is the stacked as above, (ii) $\dot{\tilde{n}}$ is a $(N_{\tilde{z}} \times N_{\tilde{n}}) \times (N_{\tilde{z}} \times N_{\tilde{n}})$ diagonal matrix with $\left.\frac{d\tilde{n}(\tilde{z},\tilde{n})}{dt}\right|_{\tilde{S}(\tilde{z},\tilde{n})\geq 0}$ as its entries, (iii) $\mathcal{X}$ is a $(N_{\tilde{z}} \times N_{\tilde{n}}) \times (N_{\tilde{z}} \times N_{\tilde{n}})$ diagonal matrix with $-\left.\frac{d\tilde{n}(\tilde{z},\tilde{n})}{dt}\right|_{\tilde{S}(\tilde{z},\tilde{n})<0}$ as its entries, and h_0 is the stacked $(N_{\tilde{z}} \times N_{\tilde{n}}) \times 1$ vector version of $\mathtt{m}_0 \pi_0(z)\,\Delta(n)$.
- The derivative matrices follow the same purpose as before, however note that $D_{\tilde{z}}$ must use a *forward difference* and $D_{\tilde{n}}$ must use upwinding to determine the approximation, depending on the sign of $\dot{\tilde{n}}$ (forward approximation for negative drift and backward approximation for negative drift).
- This expression can be rearranged to yield

$$\underbrace{\left(\underbrace{\left(-D_{\tilde{n}}\dot{\tilde{n}}\right)}_{\mathcal{N}^d} + \underbrace{\left(-\mu D_{\tilde{z}} + \frac{\sigma^2}{2} D_{zz}\right)}_{\mathcal{Z}^d} - \mathcal{X} - \delta_x \mathbf{I}_{N_{\tilde{z}} \times N_{\tilde{n}}}\right)}_{\mathbf{L}} h = -h_0. \tag{21}$$

F.3 Stationary equilibrium algorithm

- The algorithm consists of three steps, implemented in MATLAB by `SolveBEMV.m`, which is called by the master file `MAIN.m`.

Step 0 - Construct initial guess

- Start by constructing a $N_{\tilde{z}} \times N_{\tilde{n}}$ grid for log productivity and log size.
- Let $\pi(\tilde{z},\tilde{n}) = y\left(e^{\tilde{z}}, e^{\tilde{n}}\right) - e^{\tilde{n}} b - c_f$ denote the stacked $(N_{\tilde{z}} \times N_{\tilde{n}}) \times 1$ vector of flow payoffs on this grid.
- Guess an initial surplus $\tilde{S}^0$ on this grid (a $(N_{\tilde{z}} \times N_{\tilde{n}}) \times 1$ column vector); a distribution of firms over productivity and size h^0 (a $(N_{\tilde{z}} \times N_{\tilde{n}}) \times 1$ column vector); aggregate finding rates q^0 and λ^0; and an efficiency-weighted share of unemployed searchers, θ^0.
- Bundle together these aggregates $X^0 = \{q^0, \phi^0, h^0, S^0\}$.
- Construct marginal surplus. Construct exit regions, separation regions and the vacancy policy. File `InitialGuess.m` does this.
- Set $t = 1$.

Step I - Given aggregate states, iterate to convergence on the coalition's problem

- Given $X^{t-1} = \{q^{t-1}, \phi^{t-1}, h^{t-1}, S^{t-1}\}$, solve the coalition's problem to obtain S^t.
- The equation we use is
- We compute $\mathbf{B}$—which depends on the distribution of marginal surplus, and other elements of X^{t-1}—using X^{t-1} and keep it fixed. Denote this $\mathbf{B}_{t-1}$.
- Set $\tau = 0$. Set $S^{t,\tau} = S^{t-1}$. Iterate using (19), until convergence to S^{t*}, where $||S^{t,\tau} - S^{t,\tau-1}|| < \varepsilon_S$.

$$S^{t,\tau+1} = \mathbf{B}^{-1}\left(\Pi + \frac{1}{\Delta} S^{t,\tau}\right)$$

  These iterations are performed using `IndividualBehavior.m`, and the solution is assigned as the updated S^t. At each step we update $S_n^{t,\tau+1}$ using $S_n = D_n S$.

- We also obtain from the converged solution the updated separation, exit and vacancy policies.
- Note that the step size Δ cannot be too large, otherwise the problem will not converge.

Step II - Given individual behavior, iterate to convergence on aggregate states

- Given updated individual behavior in outer iteration t, obtain through iteration in an inner loop τ the distribution of firms h^t, the aggregate finding rates q^t and p^t, the share ϕ^t, the distribution of workers over marginal surplus H_n^t, the distribution of vacancies over marginal surplus H_v^t and the entry rate of firms m_0^t
- File `AggregateBehavior.m` proceeds to do this in four steps.
- Initiate each aggregate object with the previous outer iteration solution, $x^{t-1,0} = x^{t-1}$. Then:
- *Step II-a.* Update the distribution of workers over marginal surplus to $H_n^{t-1,\tau}$ and its integral $\hat{H}_n^{t-1,\tau}$ given a distribution of firms $h^{t-1,\tau-1}$ and marginal surplus S_n^t, where the latter was obtained in **Step I** above. This is done by file `Cdf_Gn.m`.
  - First consider the employment-weighted pdf:

$$h_n^{t-1,\tau}(z,n) = \frac{n}{\mathtt{n}} h^{t-1,\tau-1}(z,n)$$

    Where aggregate employment n can be normalized so that $h_n^{t-1,\tau}(z,n)$ integrates to 1.
  - We then sort h_n by marginal surplus. Note that $S_n(z,n)\, n = \widetilde{S}_{\widetilde{n}}(\widetilde{z},\widetilde{n})$, and that we solved the Bellman equation in $\widetilde{S}_{\widetilde{n}}(\widetilde{z},\widetilde{n})$, therefore we have to obtain $S_n(z,n)$ by

$$S_n(z,n) = \widetilde{S}_{\widetilde{n}}(\widetilde{z},\widetilde{n}) / n = \widetilde{S}_{\widetilde{n}}(\widetilde{z},\widetilde{n}) / e^{\widetilde{n}}.$$

  - Then sort h_n by marginal surplus and compute

$$\begin{aligned} H_n^{t-1,\tau}\left(S_n^t(z,n)\right) &= \int_0^{S_n^t(z,n)} h_n^{t-1,\tau}(s)\, ds \\ \hat{H}_n^{t-1,\tau}\left(S_n^t(z,n)\right) &= \int_0^{S_n^t(z,n)} H_n^{1-t,\tau}(s) ds \end{aligned}$$

- *Step II-b.* Update the distribution of vacancies over marginal surplus $H_v^{t-1,\tau}$ given the vacancy policy v^t, $h^{t-1,\tau-1}$, $H_n^{t-1,\tau}$, $q^{t-1,\tau-1}$, $\phi^{t-1,\tau-1}$ and the entry rate $m_0^{t-1,\tau}$. This is done by file `Cdf_Gv.m`.
  - First construct vacancies per worker for the entrants, which is the necessary amount of vacancies you need to post to get n_0 workers:

$$v_e^{t-1,\tau}(z,n) = \frac{n}{q^{t-1,\tau-1}\left[\phi^{t-1,\tau-1} + (1-\phi^{t-1,\tau-1}) H_n^{t-1,\tau}\left(S_n^t(z,n)\right)\right]} m_0^{t-1,\tau-1} \pi_0(z)\, \Delta(n)$$

  - Now consider the distribution weighted by vacancies, both of incumbents (using the policy function) and entrants:

$$h_v^{t-1,\tau}(z,n) = \frac{v^t(z,n)\, h^{t-1,\tau-1}(z,n) + v_e^{t-1,\tau}(z,n)}{\mathtt{v}}$$

    Where aggregate vacancies v can be normalized so that $h_v^{t-1,\tau}(z,n)$ integrates to 1.

– Then simply, sort h_v by marginal surplus and

$$H_v^{t-1,\tau}\left(S_n^t(z,n)\right) = \int_0^{S_n^t(z,n)} h_v^{t-1,\tau}(s)\,ds$$

- *Step II-c.* Update the distribution of firms $h^{t-1,\tau}$ and get the entry rate $m_0^{t-1,\tau}$ following the Kolmogorov forward equation in steady-state given $H_n^{t-1,\tau}$, $H_v^{t-1,\tau}$, $q^{t-1,\tau-1}$, $p^{t-1,\tau-1}$, $\phi^{t-1,\tau-1}$ and $m_0^{t-1,\tau}$. This is executed by file `Distribution.m`.

  – Using (21), this can be solved in one line.

$$h^{t-1,\tau} = -\mathbf{L}^{-1}h_0$$

  – To compute the entry rate, note that in the stationary equilibrium, the entry rate must be equal to the exit rate of firms, which can be obtained from $m_0^{t-1,\tau} = \mathbf{L}h^{t-1,\tau}$.

- *Step II-d.* Update the finding rates $q^{t-1,\tau}$, $p^{t-1,\tau}$ and share $\phi^{t-1,\tau}$ that are consistent with the vacancy policy v^t and the distribution of firms $h^{t-1,\tau}$. This is done by file `FindingRates.m`.

  – First get the total units of search efficiency in the labor market and total vacancies to construct:

$$\begin{aligned}
\mathtt{u}^{t-1,\tau} &= \bar{\mathtt{n}} - \mathtt{m}\int\int n\,dH^{t-1,\tau}(z,n) \\
\mathtt{s}^{t-1,\tau} &= \mathtt{u}^{t-1,\tau} + \xi\mathtt{u}^{t-1,\tau} \\
\phi^{t-1,\tau} &= \frac{\mathtt{u}^{t-1,\tau}}{\mathtt{s}^{t-1,\tau}}
\end{aligned}$$

  – Then, get total vacancies and use the matching function:

$$\begin{aligned}
\mathtt{v}^{t-1,\tau} &= \mathtt{m}\int\int v^t(z,n) + v_e^{t-1,\tau}(z,n)\,dH^{t-1,\tau}(z,n) \\
\theta^{t-1,\tau} &= \frac{\mathtt{v}^{t-1,\tau}}{\mathtt{s}^{t-1,\tau}} \\
q^{t-1,\tau} &= A\left(\theta^{t-1,\tau}\right)^{\beta-1} \\
p^{t-1,\tau} &= A\left(\theta^{t-1,\tau}\right)^{\beta}
\end{aligned}$$

- Iterate over the four sub-steps *Step II-a - Step II-d* until convergence and assign the updated aggregate states q^t, p^t, ϕ^t, h^t, H_n^t and H_v^t.

We subsequently iterate on **Step I - Step II** until both the surplus function and the aggregate states have converged.

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